At least almost all orbits of the Collatz map attain bounded values

…and other significant corollaries on the Syracuse problem

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28th January 2020

Abstract

This self-contained communication summarizes, simplifies and extend some theorems we have previously demonstrated. If one of its main results is that **at least almost all orbits of the Collatz map attain bounded values**, the methodology of algebraic geometry it employs, that is, the studying of how the complete binary, ternary and quaternary trees intersect each other over $\mathbb{N}$ is its most important contribution as not only does it yield very useful results but also can it be applied to a large diversity of other cases of discrete mathematics and beyond. Defining $S(a) = 2a+1$ and $V(a) = 4a+1$ we first establish a founding lemma that for any even number x the orbits of $V(x)$ and $S(V(x))$ merge and so do $S^k(V(x))$ and $S^{k+1}(V(x))$ for any even k. For any odd number y the orbits of $S(V(y))$ and $S^2(V(x))$ merge and so do $S^k(V(x))$ and $S^{k+1}(V(x))$ for any odd k. An algorithm connecting all of those merging pairs with each other will therefore solve the Syracuse problem. We here demonstrate the existence of an algorithm finitely connecting at least *almost all* of these pairs all the way back to the pair {1;3}, along with a few corollaries that are worthy of interest to manifest a final proof of the Collatz conjecture.

1 Introduction

This communication introduces a simple yet powerful methodology to study the Collatz orbits, essentially consisting of analysing each natural number with respect to its position on the complete binary, ternary and quaternary trees over $\mathbb{N}$, that is, trees defined respectively by the infinite iteration of all the possible compositions of the following operations on number 1:

- $\{\cdot 2; \cdot 2+1\}$ binary tree
- $\{\cdot 3; \cdot 3+1; \cdot 3+2\}$ ternary tree

- $\{\cdot 4; \cdot 4+1; \cdot 4+2; \cdot 4+3\}$ quaternary tree

...the binary tree thus generating an infinity of branches that has the cardinality of $\mathbb{R}$ (and the quaternary one, the cardinality of $\wp(\mathbb{R})$)[1]. For any number, defining its neighbourhood in terms of which branch (and of which length) it belongs to on each of these trees provides a framework to demonstrate very fruitful results that could actually be applied beyond the Syracuse problem, and more importantly, beyond discrete mathematics, for example in the study of the Julia sets of holomorphic functions.

2 Definitions

Note 2.1. *For all intent and purpose we will define* ***Syr(x)*** *as* ***"the next odd number in the forward Collatz orbit of*** x*".*

Whenever two numbers a and b have a common number in their orbit, we will also note $a \equiv b$, a relation that is self-evidently transitive:

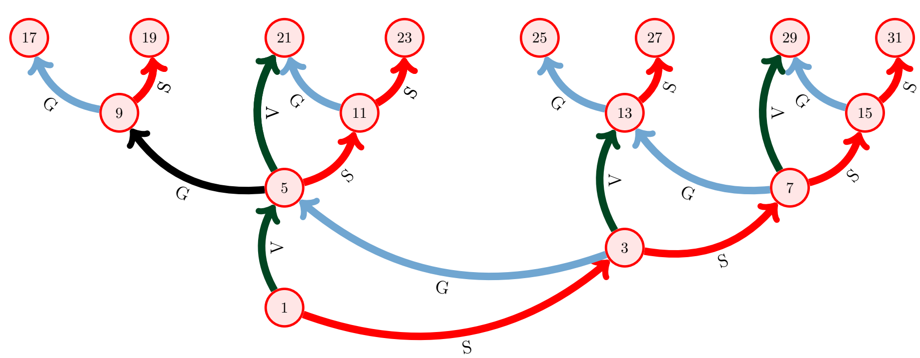
$\forall\{a; b; c\}$

$a \equiv b \wedge b \equiv c \rightarrow a \equiv c$

Definition 2.1. *Action S*
The ***Action S ("Successor")*** *on any natural number a is defined as $S(a) = 2a + 1$*

Definition 2.2. *Action V*
The ***Action V ("Vertical")*** *on any natural number a is defined as $V(a) = 4a + 1$*

Definition 2.3. *Action G*
The ***Action G ("Glacial")*** *on any natural number a is defined as $G(a) = 2a - 1$*



[1] Interestingly enough, it is because of considerations regarding the cardinality of the set of all branches of the complete ternary tree over N that the founding observations leading to this communication were made, initially over considerations regarding Feferman (2011)

Figure 1. Representation of operations V, G and S on all odd numbers from 1 to 31. The set $2\mathbb{N}+1$ is thus endowed with three unary operations without a general inverse and that are non commutative with $G \circ S = V$. Whenever we will mention the inverse of these operations, it will be assuming they exist on $\mathbb{N}$.

Definition 2.4. *Rank*
*The **rank** of an odd number is the number of **consecutive** end digits* 1 *in its base* 2 *representation, or equivalently, the number of times the action S has been applied to generate it (S is then defined on $\mathbb{N}$), and any number o of rank* 1 *can be written $S(e)$ where e is even.*

Definition 2.5. *Odd branch*
*An **odd branch** is the infinite set of numbers $\{a; S(a); S^2(a)\ldots\}$ where a is of rank 1.*

Definition 2.6. *Glacis*
*The **glacis** of bottom b is the infinite set of numbers $\{G(V(b)); G^2(V(b))\ldots\}$ where b is any odd number.*

Definition 2.7. *Root*
*The **root** r of any odd branch is its only number of rank 1.*

Definition 2.8. *Determinant*
*The **determinant** d of any odd branch of root r is $G^{-1}(r)$. Any number of rank* 1 *admits a determinant.*

Definition 2.9. *Vertical, Verticality*
A number a of rank 1 *always admits a $V^{-1}(a)$ which is called its **Vertical**. To avoid any confusion, when ambiguous, we will call $V(a)$ the V^+ of a. If the vertical of a is odd, we will call it **Vertical odd**, otherwise, it is **Vertical even**.*
*The **Verticality** of a number a of rank* 1 *is the vector $\begin{bmatrix} n \\ b \end{bmatrix}$ where b is either an even number or a number of rank* 2 *or more, and $a = V^n(b)$ We will say that a has a verticality of n and of bottom b.*

Definition 2.10. *Successal, Successality*
A number a of rank 2 *or more will be called **Successal**, and its **successality** is equal to its rank.*

Definition 2.11. *Glaciality*
*In a glacis of bottom b, the **glaciality** of $S(b)$ is set to* -1*, that of $V(b)$ is set to* 0 *and that of $G^n(V(b))$ is set to n. To aggregate the information of the bottom b of its glacis to any glacial number (that is, a number that can be written as $G(x)$ where x is odd), its glaciality will be the vector $\begin{bmatrix} n \\ b \end{bmatrix}$*

3 Essential lemmas

Lemma 3.1. *If $a = V(b)$ and b is odd, then $Syr(a) = (Syr(b))$ and we will note $a \equiv b$*

Proof. If a is written $4b+1$ then $3a+1 = 12b+4 = 4(3b+1)$ therefore $a \equiv b$

$\square$

This lemma is quite trivial and therefore in no way original, but it is an essential building block nonetheless.

Lemma 3.2. *Let a be a number of rank 1 and of determinant d, then $Syr(S(a)) = G(3 \cdot d)$ Let a be a number of rank n in an odd branch of determinant d, then $Syr^{n-1}(a) = G(3^{n-1} \cdot d)$*

Proof. If a is of determinant d then $a = 2d-1$, and of course d is odd.

$S(a) = 4d - 1$

$\frac{3S(a)+1}{2} = \frac{12d-2}{2} = 6d - 1 = G(3 \cdot d)$

$Syr(S(a)) = G(3 \cdot d)$

Now let's generalize to the n

Note that if $Syr(S(a))$ can be written $G(3 \cdot d)$ it is also of rank 1, whereas S(a) was of rank 2, therefore, the action of Syracuse has made it lose one rank.

All we thus have to prove now is that $Syr(S^2(a)) = S(Syr(S(a)))$ under those conditions

$\frac{3 \cdot (S^2(a))+1}{2} = 6a + 5$

$S(Syr(S(a))) = S(3a+2) = 6a + 5 = Syr(S^2(a))$ $\square$

Corollary 3.3. *If a is of rank $n > 1$, Syr(a) is of rank n-1, and Syr(S(a)) = S(Syr(a))*

Note 3.4. *The Syracuse action over an odd number is equivalent to adding 1 to it, then the half of the result, then -1. How many times one can add an half to the number $+1$ directly depends on the length of the immediate even branch of the binary tree that is to its right.*

Let us take Mersenne numbers for example, that are defined as $2^n - 1$. One can Syracuse them consecutively a number of time that is proportionate to their rank-1, indeed, 31, which is written 11111 is of rank 5, because $32 = 2^5$ so if you repeat the action "add to *the number the half of itself" which is equivalent to a multiplication by $\frac{3}{2}$ this will yield an even result* ***exactly four consecutive times****.*

Thus, any ascending orbit in Syracuse concerns only numbers a of rank 2 or more, and is defined by

$$(a+1)\cdot(\tfrac{3}{2})^{rank-1}-1$$

because the rank is strictly equivalent to the length of the next even branch of the binary tree on the right of the number, defining how many consecutive times the action $\cdot\frac{3}{2}$ will yield an even number.

Lemma 3.5. *Let a be an odd number of rank 1 that is vertical even, then $3a$ is successal, and $9a$ is vertical even.*
Let a be an odd number of rank 1 that is vertical odd, then $3a$ is successal, and $9a$ is vertical odd.

Proof. If a is vertical even it can be written $8k+1$ $\forall k$

$3a = 24k+3$ and this number admits an S^{-1} that is

$12k+1$, which is an odd number, therefore $3a$ is Successal

$9a = 72k+9$ and this number admits a V^{-1} that is

$18k+2$, an even number indeed.

Now if a is vertical odd, it can be written $8k+5$ $\forall k$

$3a = 24k+15$ and $9a = 72k+45$, $3a$ admits an S^{-1} and $9a$ admits a V^{-1}, respectively

$12k+7$ and $18k+11$ and they are both odd. □

Theorem 3.6. *(regular quaternary equivalence)*
Let a be a number that is vertical even, then $(a) \equiv S(a)$ and $S^k(a) \equiv S^{k+1}(a)$ for any even k. Let a be a number that is vertical odd, then $S(a) \equiv S^2(a)$ and $S^k(a) \equiv S^{k+1}(a)$ for any odd k.

We will call these relations merging alternate pairs of odd branches ***"regular quaternary equivalences"*** *or* ***qe****.*

Proof. If a is vertical even then it can be written as $G(d)$ where d is necessarily vertical (odd or even)

so by lemma 3.5 we have that $3d$ is successal and by lemma 3.2 we have $Syr(S(a)) = G(3d)$ so it is necessarily vertical odd (since $3d$ is successal) so $Syr(a) = V^{-1}(Syr(S(a))$ and therefore $a \equiv S(a)$

This behavior we can now generalize to the n, because if a is vertical even and of determinant d, then the lemmas we used also provide that $Syr^n(S^n(a)) = G(3^n \cdot d)$ and therefore $Syr^n(S^n(a))$ will be vertical even for any even n because $3^n \cdot d$ will be vertical something (even or odd, depending on what the determinant was) for any even n.

Now if a is vertical odd it can be written $G(d)$ and d is necessarily successal because $G \circ S = V$.

Thus $3d$ is vertical (even or odd) and therefore $Syr(S(a)) = G(3d)$ is vertical even. $\square$

This **qe theorem** is a more elaborate, and now very useful building block of our demonstration, because it allows to place a relation of equivalence between every other pair of any odd branch of the binary tree, up to infinity. It is also based on the characteristics of increasing phases of the Syracuse orbits: any number of rank n is finitely turned into a number that has a vertical, which is either odd or even.

We should now interest ourselves with the decreasing phases of the Syracuse orbits [2], and they only concern the glacis.

Theorem 3.7. *(glacial decreasing)*
Let a be a vertical even number with a glaciality of $\begin{bmatrix} n \\ b \end{bmatrix}$ *where n is even, then* $a \equiv 3^{\frac{n}{2}}(b)$
Let a be a vertical even number with a glaciality of $\begin{bmatrix} m \\ b \end{bmatrix}$ *where m is odd, then* $a \equiv S(3^{\frac{m+1}{2}}(b))$

Proof. if a is of glaciality $\begin{bmatrix} n \\ b \end{bmatrix}$

then by definition $a = 2^{n+2}b + 1$.

Then $3 \cdot a + 1 = 3(2^{n+2}b + 1) + 1) = 2^{n+2} \cdot (3b) + 4$.

As this expression can be divided by no more than 4, we have

$Syr(a) = 2^n 3b + 1$, therefore the glaciality of $Syr(a)$ is

$$\begin{bmatrix} n-2 \\ 3b \end{bmatrix}$$

Note that if $n = 2$ then $V^-(Syr(a)) = V^-(2^2 \cdot (3a) + 1) = 2^2 \cdot \frac{1}{4} \cdot (3b) = 3b$ which is of course an odd number. Therefore $Syr(a)$ is vertical odd and $V^-(Syr(a)) = 3b$ thus we have proven that $a \equiv 3b$.

[2]remember that we are still only considering odd numbers when we write "decreasing phases", and still defining $Syr(a)$ as "the next odd in the orbit of a"

If $n = 1$ then $a = 2^3 \cdot b + 1$ so $3(a+1) = 2^3 \cdot 3b + 4$ therefore $Syr(a) = S(3b)$ and thus $a \equiv S(3b)$.

From this we can generalise the progression of glacis numbers. Let b be any odd number, thus defining a glacios bottom. All **"Variety S"** numbers of its glacis are written $V(b \cdot 2^{2k-1})$ or $S(b \cdot 2^{2k}) = 2^{2k+1} \cdot b + 1$ and all **"Variety V"** numbers of its glacis are written $V(b \cdot 4^k)$ or equivalently $S(b \cdot 2^{2k+1}) = 4^{k+1} \cdot b + 1$. Any glacis number g of order $2k$ or $2k-1$ may be thus reduced to a glaciality of 0 or -1 by the following transformation: $(g-1) \cdot \left(\frac{3}{4}\right)^k + 1$ therefore we do have indeed that,

- for Variety S numbers : $2^{2k+1} \cdot b \cdot \left(\frac{3}{4}\right)^k + 1 = 2b \cdot 3^k + 1 = S(b \cdot 3^k)$
- for Variety V numbers : $4 \cdot 4^k \cdot b \cdot \left(\frac{3}{4}\right)^k + 1 = 4b \cdot 3^k + 1 = V(b \cdot 3^k)$

As the obtaining of these equalities will fit any odd number b, we have that any glacis number of glaciality $2k$ will be finitely mapped to $4b \cdot 3^k + 1 = V(b \cdot 3^k)$ and that any glacis number of order $2k-1$ will be mapped to $2b \cdot 3^k + 1 = S(b \cdot 3^k)$.. **Any glacis number merges either directly with a power of three of its bottom or with the successor of it.** $\square$

4 General description of the Syracuse dynamic

The **qe theorem** allows to place an infinite amount of equivalences along the binary tree that are the result of quaternary properties.

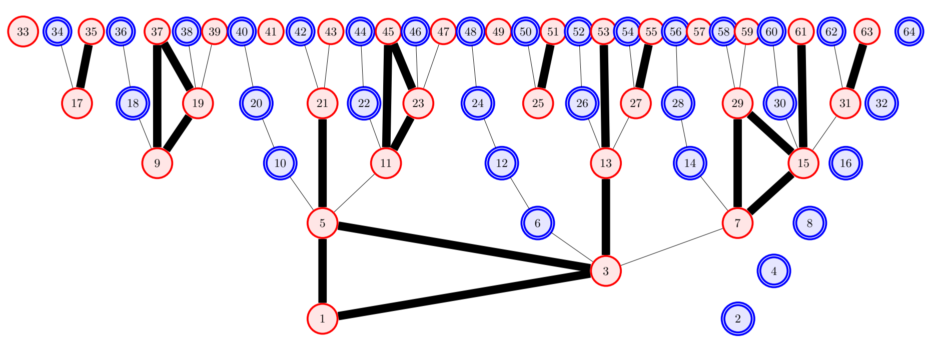


Figure 2. A representation of the intersection of the binary and the quaternary trees: as only the quaternary operation $\cdot 4 + 1$ really matters to the representation, we have just

warped the binary tree so that V(a) is indeed *Vertical* to a. We have also not connected the rank 1 numbers to the even ones of which they are the successors so as to make glacis more easily visible. The bold black lines indicate **qe** equivalences and vertical equivalences, and so whenever numbers are joined by a connective series of those lines, their Collatz orbits merge. Connecting all of those equivalences together completely solves the Syracuse problem, and as we will see in the next section, this endeavour requires the introduction of a third dimension: that of the ternary representation of any number.

4.1 Ascending phases

The orbit of an odd number can only increase if it is of rank 2 or more. Odd branch numbers of rank $n > 2$ ascend with the progression $+1 \cdot \frac{3}{2}^{n-1} - 1$ and this allows to compress their orbit to the next glacis.

More particularly though, if a pair is not connected by the **qe** equivalence, **it is because the rank 1 reduction of its smallest number by Syracuse is vertical odd**. When we have a pair $\{a; S(a)\}$ where a is vertical odd, what happens is that a is mapped to $Syr(V^{-1}(a))$ and $S(a)$ is mapped to the glacis of bottom $Syr(V^{-1}(a))$ and this phenomenon, by which a pair of numbers that were related on the binary tree become separated we call an **Avalanche**. Avalanches account for absolutely all of the chaoticity of the Syracuse orbits, and whoever can perfectly predict their occurence and consequences has cracked *all* of the Syracuse orbits. Needless to say, this communication doesn't have this ambition.

An example of Avalanche can be understood on **figure 2** by observing the pair $\{3; 7\}$ of which the first Syracuse image is $\{5; 11\}$ where 5 indeed is vertical odd. The avalanche is that 5 is mapped to the image of 1, which is 1, and 11 is mapped to 17, which is in the glacis of bottom 1. This happens to all such pairs, which we have called ”buds” in a previous work.

The alacrious reader will not fail to notice that 17 is precisely a glacis number of Variety V, and this is not happening by chance: if a vertical odd number a is the finite vertical of an even number, then $S(a)$ will be mapped as a variety V in the next glacis, and proportionally as high as a was vertical, and if the bottom of its verticality is odd, then $S(a)$ will be mapped as a variety S. The easy reason for which it is so will be left for the reader to grasp, as we do not use it in the other demonstrations of this piece.

We have also noted that any power of 9 of a vertical number is either of the series ”vertical odd” or the series ”vertical even” and these two are parallel: one cannot obtain a vertical odd number by applying any power of nine to a vertical even one. Thus, the destiny of branches the determinant of which falls within the ”vertical odd” or ”vertical even” series is quite different, and this level of precision can also help understand the Syracuse dynamic better. Still, we shan't use it further in this communication, even though it is more significant than a simple curiosity.

4.2 Descending phases

Any odd number can only decrease in *Syr* if it is of rank 1, and then it intersects either the Successor or the Vertical of a power of 3 of the bottom of its glacis. Note that in so doing, it always encounters the consequences of another **qe** on the branch it meets: either the power of three of the bottom of the glacis can be proven to merge with its successor, either it is vertical odd or it is its predecessor that is merging with it.

Here too, powers of 3 of the bottom of the glacis, just as being the case with odd branch determinants had useful implications to elaborate more advanced theorems, can either be of the "vertical odd" or "vertical even" sequences.

Numbers of even glaciality n decrease following the dynamic $-1 \cdot (\frac{3}{4})^{\frac{n}{2}} + 1$ and those of odd glaciality m, $-1 \cdot (\frac{3}{4})^{\frac{m+1}{2}} + 1$.

Thus, **whenever one can prove that an odd number merges with its triple, one proves that it merges with its first number of even glaciality**

Also, **whenever one can prove that an odd number merges with the successor of its triple, one proves that it merges with its first number of odd glaciality.**

We cannot stress enough the importance of this pair of results.

5 Using the ternary tree to connect the quaternary equivalences

5.1 Definitions

In the previous sections we have mostly identified numbers by their position in the complete binary and quaternary trees over $\mathbb{N}$. Using statements of the kind "a is vertical even" is a typical example of crossing binary and quaternary properties to identify specific characteristics of a number. We will now expand this methodology by adding the ternary tree, which elements we will identify with the following definitions:

Definition 5.1. *Ternary, Ternarity*
*A number b is **ternary** or of **type B**, if it can be divided by 3. Its **ternarity** is the total number of times it can be divided by 3, to which we will add the information of the non-ternary number resulting from this finite operation, thus the full ternarity of any ternary number that can be written $3^n \cdot x$ where x is non ternary is $\begin{bmatrix} n \\ x \end{bmatrix}$ For all intent and purpose, when we will refer to just the "ternarity" of a B type number (as opposed to "full ternarity") we will just be meaning n alone.*

Definition 5.2. *1-ternary, 1-ternarity*
*A number c is **1-ternary** or of **type C**, if its base 3 representation ends with digit 1. The number of times one can remove a consecutive end digit 1 (we call this operation C^{-1}) is its **1-ternarity**, to which we add the information of the number resulting from it. Thus, a number c that can be written $x\underbrace{1\dots1}_{n}$ in base 3 has a 1-ternarity of $\begin{bmatrix} n \\ x \end{bmatrix}$*

Definition 5.3. *2-ternary, 2-ternarity*
*A number a is **2-ternary** or of **type A**, if its base 3 representation ends with digit 2. The number of consecutive times one can remove an end digit 2 (we call this operation A^{-1}) is its **2-ternarity**, to which we add the number resulting from it. Thus, a number a that can be written $x\underbrace{2\dots2}_{n}$ in base 3 has a 2-ternarity of $\begin{bmatrix} n \\ x \end{bmatrix}$*

Definition 5.4. *"up", "down"*

*A number is called **"up"** if its **qe** makes it merge with its successor. If $B \equiv S(B)$ we call it a **Bup** and respectively for A and C, **Aup** and **Cup**. If $B \equiv S(B)$, as we necessarily have that $S(B)$ is of type C, we will call this C "down" or Cdown. If a number is vertical odd, it is "down", if it is vertical even, it is "up".*

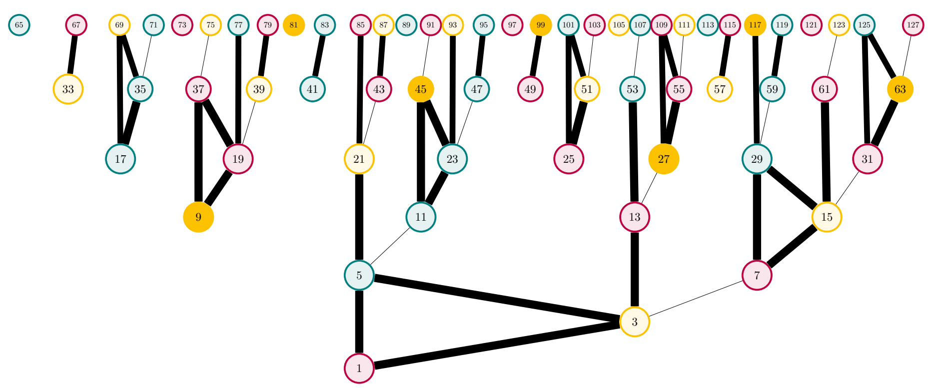


Figure 3. All odd numbers from 2^0 to 2^7. Type A are circled in teal, B in gold and C in purple. Numbers of a ternarity of 2 or more (numbers that can be divided by 9, that is) are also colored in gold. To explain again the previous definitions: 27 is a *Bup* and 63 is a *Bdown* for example. Any glacis number of glaciality above 0 is "up": 17 is an *Aup* for example, and 19 a *Cdown*. Though exotic, these names are absolutely essential to the results we obtain, and their very use is a pure result of our methodology: the "up" and "down" properties come from the study of the intersections of the binary and quaternary trees, and the A,B,C ones, from the ternary.

5.2 Golden Automaton

We shall now define an algorithm that can mend together infinitely many quaternary equivalences, and actually **at least almost all of them**. This algorithm we call the **Golden Automaton** or **GA**.

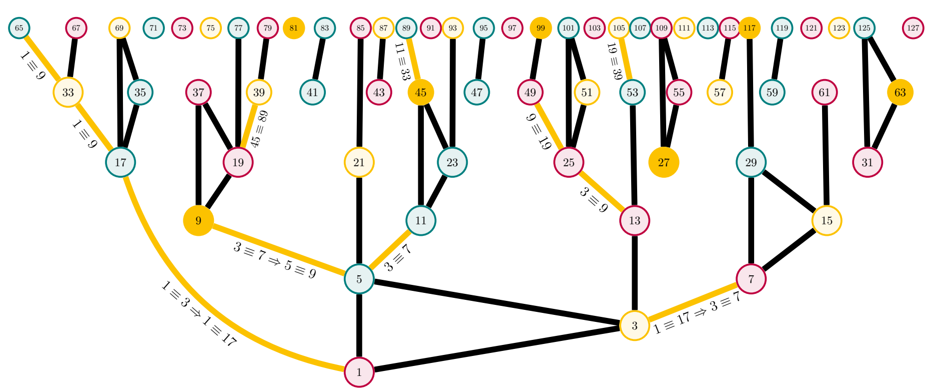
1. Start from the equivalence $1 \equiv 3 \equiv 5$
2. Whenever a new type B number is found in the network, divide it by 3, and always prioritize the Bups

   for the first round, the only B is 3, so the GA notes that $1 \equiv 3$
3. project the equivalences in the corresponding glacis

   at round 1, there is only one equivalence: having that 1 *merges with its triple means it also merges with its first glacis number of the V Variety, thus giving* $17 \equiv 1$
4. whenever a new type A glacis number is found in the network, apply A^{-1} up to a non A type[3], which is added to the network. If this generates an equivalence of the type $B \equiv S(B)$, prioritize it.

   at round 1, this gives $17 \equiv 7$*. Also by the* ***qe*** $7 \equiv S(15)$ *and* 15 *is of type B, but not a Bup. so it will not be prioritized by the GA. Still, we now have* $3 \equiv S(3)$*, which will be put to use in the glacis of bottom* 1
5. project the equivalence in the corresponding glacis

   $3 \equiv S(3) \rightarrow 1 \equiv 9$. 9 *is ternary, and being a glacis number, it is a Bup, so it will now prove two glacis equivalences together, namely those of* 49 *and* 35 *(each of them bringing two Bdowns) and equivalently,* 65 *and* 33.
6. After checking the *Aups* and prioritizing those of the highest 2-ternarity (they are easy to spot, as they can be written $G(B)$ where B is of high ternarity) check all the *Bdowns* added to the network, including the ones that are the verticals of an A type. Always prioritize the lowest numbers in any case.
7. repeat.



[3] This operation can only be finitely repeated of course, and is decreasing

Figure 4. The golden branches on this figure are only a *subset* (and precisely: the *Bup* series after exploiting 3) of the equivalences the GA proves to reach them: assuming number 3 already grafted to the network (since we start from $1 \equiv 3 \equiv 5$), they are the ones generated by only considering the *Bups*, and not the (many) more opportunities offered by connecting *Bdowns*, which are of lesser immediate interest but of much higher frequency and actually always yield either a new *Aup* or a new *Bup* to graft as well.

Theorem 5.1. *The Golden Automaton can neither stop nor loop.*

Proof. The moment a *Bup* is brought to its growing equivalence network, the GA can finitely prove that a number of either type A or C merges with two consecutive glacis numbers. Also, if the *Bup* is of a ternarity above 1, it will additionally prove that two glacial type C merge with a type B bottom, for example $1 \equiv 9$ means $65 \equiv 33 \equiv 1$ but also $49 \equiv 25 \equiv 3$ because 9 is of ternarity 2. Exploiting the first *Bup*, there will always be an *Aup* and *Bup* that will be freshly grafted to the equivalence network, either in the descending glacis order $(B; A)$ if the bottom is of type C, and $(A; B)$ if it is of type A. The more ternary the *Bup*, the higher-reaching the equivalence. That a *Bup* be brought in the network guarantees a full new equivalence of the type $x \equiv 3^k x \equiv S(3^k)$ which will always reconnect a new *Aup* and a new *Bup*. Thus, the grafting of a new *Bup* to the network always implies the grafting of another one. So it cannot stop.

Can it loop now? Any *Bup* the GA gets, it does by either operations $G \circ G \circ V$ or $G \circ V$, and then it will also get an *Aup* in the process. The equivalence it obtains is then that $\frac{Bup}{3}$ merges both with $G \circ G \circ V(\frac{Bup}{3})$ and $G \circ V(\frac{Bup}{3})$ this progression is strictly increasing, precisely because glacis numbers have a strictly decreasing progression in Syracuse. □

Theorem 5.2. *The Golden Automaton connects at least almost all orbits.*

Proof. So we have an algorithm that never stops, cannot loop and that recoups an exponentially growing diversity of numbers within the ternary tree over $\mathbb{N}$. Also, **all** the ascending Syracuse orbits lead to numbers that can be written $G(B)$, and **all** the descending ones, to numbers that are either B or $S(B)$. Remember too, that, regarding the title of this communication, whenever an orbit is grafted by the GA, it is never *almost* bounded, but bounded, period.

Besides, we have not at all fully exploited all the *Bdown* the GA grafts to its equivalence network, and which new equivalences they also prove. Even though each of them is initially less powerful than the grafting a *Bup* (*ie.* in its first round of exploitation, a freshly grafted *Bdown* either offers the grafting of a new *Aup* or that of a new *Bup*) each round of running the GA brings proportionally much more of them, and this for simple reasons kind

- whenever the GA grafts a number of glaciality above 2, it also grafts its next glacial image, which is necessarily of type C. For example, as we have already seen, grafting 65 and 33 implies the grafting of the type C 49 and 25 (their next glacial images) and therefore that of the *Bdowns* 99 and 51 which are their successors, which which they merge by virtue of the **qe**.

- whenever the GA grafts a type A number, each application of A^{-1} produces either another A type which vertical is then a *Bdown*, or if it is of a 2-ternarity of 1, directly either a *Cup* or a *Bup*. As any glacial type A can be written G(B), the ternarity of its determinant gives the number of times A^{-1} can be applied (and so how many new *Bdowns* it gives, minus one). For example: 17 being the glacial image of 7, and $G(9)$, A^{-1} can be applied only twice to it, so only one type A number is also grafted (11 that is) thus grafting $V(11)$ which is 45.

Thus, *Bdown* that is grafted to the ternary tree (along with its third) always initiates the adding *at least* of an endless other series of *Bdowns* and of *Bups* too, this also, for elementary algebraic reasons:

- Any freshly grafted *Bdown* allows the grafting of either a glacis A or a glacis B depending on whether this *Bdown* can be written $3^n \cdot c$ (then it grafts an *Aup*) or $3^n \cdot a$ (then it grafts a *Bup*).
- if an *Aup* is thus grafted, it finitely connects either a *Bup* or a *Cup* and therefore a *Bdown* (as in what happens when connecting the pair $7 \equiv 15$), because this *Aup*, in base 3, can only be written either $b\underbrace{2\ldots2}_{n}$ (in which case it finitely maps to a *Bup*) or $c\underbrace{2\ldots2}_{n}$ (in which case it finitely maps to a *Cup*). Thus the play again through the *Bdown* series is guaranteed for the Golden Automaton.
- if it's a *Bup*, it now grafts *both* a glacis A *and* a glacis B, the play again is also guaranteed.

Thus the GA does not stop, both through the grafting series that started with the first *Bup* (namely 9, then 33) but also to the one that started through the many *Bdowns* **after** it (15, 21, 45...). The existence of the **qe** also proves that the only problems that need to be solved to *fully* demonstrate that all Collatz orbits are bounded are the A-A avalanches, that is, the demonstration that all glacis numbers of type A merge with their glacis Bottom, which would allow to prove that all the numbers of any odd branch merge. The Golden Automaton not only guarantees that an infinite amount of those A-A avalanches are solved but also that, for the progression of it to 3^n, when n goes to infinity, *at least* **almost all** are solved. $\square$

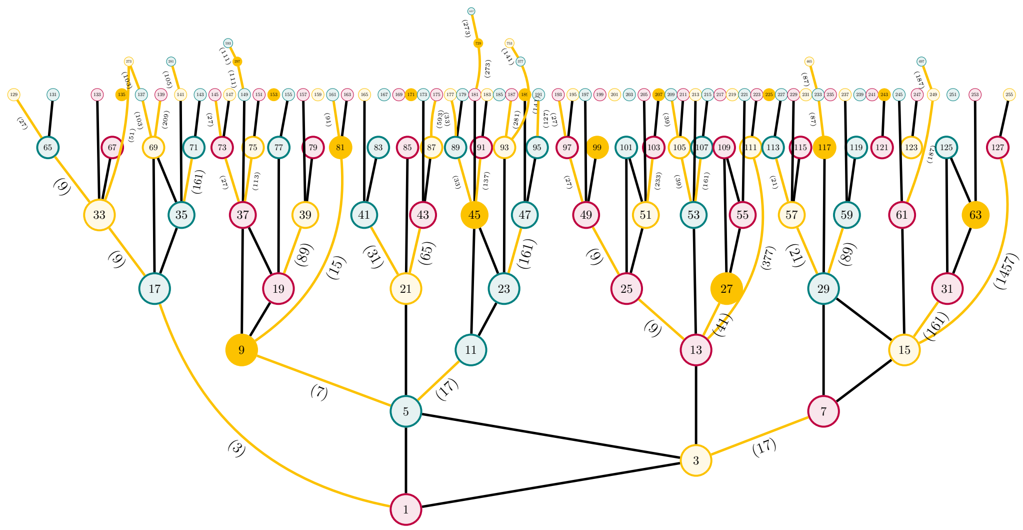


Figure 5. Note that the development of the Golden Automaton mimics the one of the famous "Collatz Seaweed", which is of course no accident. This more complete description of the Automaton, grafting the **qe** together as it progresses from $1 \equiv 3 \equiv 5$, goes all the way to grafting 127 and 31. Each branch indicates which number generates it, for example 9 is grafted to 5 because 7 is grafted to 3, and 9 being a *Bup* it grafts 33 and 65. **Never before has such a large proportion of any odd numbers**[4] **up to to $2^n + 5$ been demonstrated to finitely merge with 1, and with an algorithm that has of course nothing to do with the brute force computation of each orbits, in that not only are each of its steps finite and predictable, but the demonstrating that all numbers from 2^0 to $2^n + 5$ converge has not the algorithmic complexity of calculating each single orbits**. In this figure are shown the grafting of 31 and 127, thus, even though all the consequences of this grafting are not represented, proving the merging of all numbers up to 261.

The most problematic numbers at any stage 2^n remain the Mersenne numbers and of course, the ones leading to them, yet the Golden Automaton grafts them very easily. All the problematic Mersennes are *Cups* like 31, 127 and the like. Since the determinant of the Mersenne branch is 1, at any level of the binary tree, their first glacial image will always be written $G(3^n)$. For example, the first forward glacial image of 31 is $G(81) = 161$. This is the reason 31, for example, cannot cycle when reaching its first glacis: being of high rank, it has to be written as the glacial of a proportionately high power of 3 of its determinant (1), which just can never be a vertical of 1 (no power of 3 of 1 can be written $V^n(1)$). Also, since multiplying a number of a high verticality by 3 always gives a number of proportionate successality, it means that the precursors of the Mersenne *Cups* will always be written $G(V^n(1))$, for example the precursor of 7 is 9 and that of 31 is $G(21) = 41$. Logically, the precursor of 127 is $G(85) = 169$, that of 511 is

[4] because it is easy to prove that if all numbers up to $2^n - 1$ converge then so do all odds until at least $2^n + 5$. Indeed, $2^n + 1$ being glacial, always have a smaller number than $2^n - 1$ in its orbit, from n=4 onward, $2^n + 3$ is always the successal of a glacial and $2^n + 5$ is always a vertical odd number

$G(341) = 611$.

Note that it is equally easy to demonstrate that the precursors of the "Anti-Mersenne" of type A, that is, the type A numbers that can be written $2^n + 1$ or equivalently the type A numbers of the glacis of bottom 1, can always be written $S(V^n(1))$: 11 leads to 17, 43 leads to 65, 171 leads to 257 etc.

As $729 = 3^6$ is of Variety S, $G(729) \equiv 127$ is of Variety V, which makes it much easier to solve as it brings its solving to the problem of grafting $91 \cdot 3 = 273$, which is itself of Variety V. This is why 127 is proven to merge with 15 before we even prove that 31 converges as well.

31 indeed is a much harder number to graft abstractly because its first glacial image is of Variety S, meaning it requires the grafting of many more ternary numbers and their successors to achieve its grafting, but it works nevertheless. It is easy to prove that if the first glacial image of 31 is of Variety S, then that of 127 is of Variety V, but that of 511 will be of variety S too. Hence: that 31 is "long" to solve means 127 is "short", and thus 511 is "long" but 2047 is "short". 31 was "long" also because 7 was "short": 9 is of variety S, hence the first glacial image of 7 is of variety V, and easy to graft back to 1.

One can very easily infer from **Figure 5** that with the infinity of avalanches the Golden Automaton solves, including those of the dreaded Mersenne numbers, *at least* ***almost all*** orbits attain, not *almost bounded* but **bounded** values, period. The emphasis on *at least* means it could be proven, through more abstract methods, that no Collatz Orbit can escape a finite development of the Golden Automaton, a statement this article will leave as a conjecture for now.

Conjecture 5.3. *For any Collatz orbit there is an arbitrarily long, but finite, development of the Golden Automaton that grafts it back to 1.*

6 Conclusion

Studying the intersections of the complete binary, quaternary and ternary trees over $\mathbb{N}$ yields simple yet very powerful results regarding the Syracuse dynamic. As we have already pointed out in Aberkane (2020) this methodology could be extended to other discrete dynamic systems, diophantine problems and even to problems involving non-discrete environments.

Does the Golden Automaton's exponential grafting of new equivalences allow to demonstrate that actually *all* the Collatz orbits are finitely grafted to it? This will be left to another document although the curious reader may very well establish their own rigorous demonstration in that sense already.[5]

[5] The author encourages the ever alacrious reader to verify it by a method similar to Heule *et al.* (2016)'s "cube and conquer".

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