Endomorphisms of the Collatz quiver

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Abstract

We analyses the $3n+1$ dynamical system in terms of both the fundamental and emerging properties of the quivers it generates over $2\mathbb{N}+1$. In particular, we offer a first description of the endomorphism underlying the structure of the quiver known among popular mathematics circles as the "Collatz Seaweed" and describe its generative rule. We then establish some fundamental properties of other endomorphisms generated by the $3n+1$ dynamical system, mapping this quiver onto proper subsets of $2\mathbb{N}+1$ composed of the nodes we prove any Collatz orbit must intersect (which we call "Determinants") and draw interesting conclusions from those properties, allowing us to claim a result strictly superior to the one of Tao (2019) on the Collatz conjecture.

1 Introduction

The dynamical system generated by the $3n+1$ problem is known to create complex quivers over $\mathbb{N}$, one of the most picturesque being the co-called "Collatz Seaweed", a name popularized by Clojure programmer Oliver Caldwell in 2017. In this research we bring new levels of detail in the characterisation of these oriented graphs and will give the first ever description of the underlying generative rules that allow to reconstruct the "seaweed". Our methodology first consists of covering $\mathbb{N}$ with an *ad hoc* coordinate system the geodesics of which are the branches of the complete binary, ternary and quaternary trees developed over the Natural numbers, that is, the iterations of the following three sets of actions, for each of them, in any order

- $\{\cdot 2; \cdot 2+1\}$ binary tree
- $\{\cdot 3; \cdot 3+1; \cdot 3+2\}$ ternary tree
- $\{\cdot 4; \cdot 4+1; \cdot 4+2; \cdot 4+3\}$ quaternary tree

The binary tree over $\mathbb{N}$ thus generates an infinity of branches that has the cardinality of $\mathbb{R}$.

1.1 Definitions

Note 1.1. *For all intent and purpose we will define **Syr(x)** or the **"Syracuse action"** as **"the next odd number in the forward Collatz orbit of** x**"**.*

Whenever two numbers a and b have a common number in their orbit, we will also note $a \equiv b$*, a relation that is self-evidently transitive:*

$\forall\{a; b; c\}$

$a \equiv b \wedge b \equiv c \rightarrow a \equiv c$

Definition 1.1. *Action* S
*The **Action S ("Successor")** on any natural number* a *is defined as* $S(a) = 2a + 1$

Definition 1.2. *Action* V
*The **Action V ("Vertical")** on any natural number* a *is defined as* $V(a) = 4a + 1$

Definition 1.3. *Action* G
*The **Action G ("Glacial")** on any natural number* a *is defined as* $G(a) = 2a - 1$

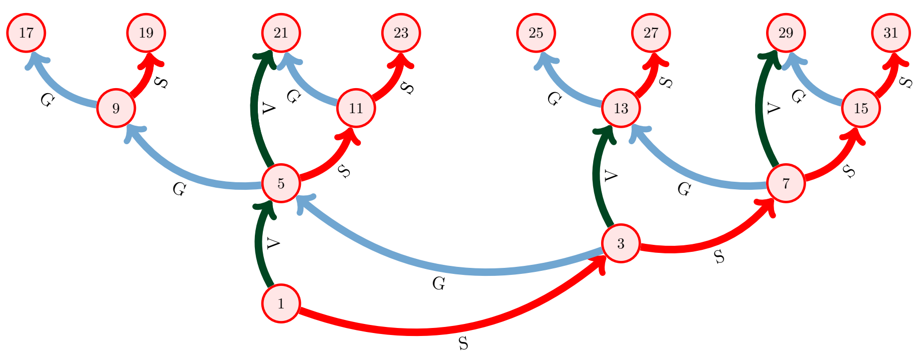

Figure 1. Quiver connecting all odd numbers from 1 to 31 with the arrows of operations S,V and G. The set $2\mathbb{N}+1$ is thus endowed with three unary operations without a general inverse that are non commutative with $G \circ S = V$. Whenever we will mention the inverse of these operations, it will be assuming they exist on $\mathbb{N}$.

Definition 1.4. *Rank*
*The **rank** of an odd number is the number of **consecutive** end digits* 1 *in its base* 2 *representation, or equivalently, the number of times the action* S *has been applied to generate it (*S *is then defined on* $\mathbb{N}$*), and any odd number* o *of rank* 1 *can be written* $S(e)$ *where* e *is even.*

Definition 1.5. *Odd branch*
An ***odd branch of root r*** *is the infinite set of numbers* $\{r; S(r); S^2(r)\dots\}$ *where* r *is of rank 1.*

Definition 1.6. *Glacis*
The ***glacis of bottom b*** *is the infinite set of numbers* $\{G(V(b)); G^2(V(b))\dots\}$ *where* b *is any odd number.*

Definition 1.7. *Pivot*
The ***pivot p*** *of any odd branch of root* r *is* $G^{-1}(r)$*. Any number of rank* 1 *admits a pivot.*

Definition 1.8. *Vertical, Verticality*
A number a *of rank* 1 *always admits a* $V^{-1}(a)$ *which is called its* ***Vertical****. To avoid any confusion, when ambiguous, we will call* $V(a)$ *the* V^+ *of* a*. If the vertical of* a *is odd, we will call it* ***Vertical odd****, otherwise, it is* ***Vertical even****.*
The ***Verticality*** *of a number* a *of rank* 1 *is the vector* $\begin{bmatrix} n \\ b \end{bmatrix}$ *where* b *is either an even number or a number of rank* 2 *or more, and* $a = V^n(b)$ *We will say that* a *has a verticality of* n *and of bottom* b*.*

Definition 1.9. *Successal, Successality*
A number a *of rank* 2 *or more will be called* ***Successal****, and its* ***successality*** *is equal to its rank.*

Definition 1.10. *Glaciality*
In a glacis of bottom b, the ***glaciality*** *of S(b) is set to* -1*, that of V(b) is set to* 0 *and that of* $G^n(V(b))$ *is set to* n*. To aggregate the information of the bottom* b *of its glacis to any glacial number (that is, a number that can be written as* $G(x)$ *where* x *is odd), its glaciality will be the vector* $\begin{bmatrix} n \\ b \end{bmatrix}$

2 Essential lemmas

Lemma 2.1. *If* $a = V(b)$ *and* b *is odd, then* $Syr(a) = (Syr(b))$ *and we will note* $a \equiv b$

Proof. If a is written $4b + 1$ then $3a + 1 = 12b + 4 = 4(3b + 1)$ therefore $a \equiv b$

□

This lemma is quite trivial and thus in no way original, but it is an essential building block nonetheless.

Lemma 2.2. *Let* a *be a number of rank* 1 *and of pivot* p*, then* $Syr(S(a)) = G(3 \cdot p)$ *Let* a *be a number of rank* n *in an odd branch of pivot* p*, then* $Syr^{n-1}(a) = G(3^{n-1} \cdot p)$

Proof. If a is of pivot p then $a = 2p - 1$, thus p is odd.

$S(a) = 4p - 1$

$\frac{3S(a)+1}{2} = \frac{12p-2}{2} = 6p - 1 = G(3 \cdot p)$

$Syr(S(a)) = G(3 \cdot p)$

Let's generalize to the n

Note that if $Syr(S(a))$ can be written $G(3 \cdot p)$ it is also of rank 1, whereas $S(a)$ was of rank 2, therefore, the action of Syracuse has made it lose one rank.

All we thus have to prove now is that $Syr(S^2(a)) = S(Syr(S(a)))$ under those conditions

$\frac{3 \cdot (S^2(a))+1}{2} = 6a + 5$

$S(Syr(S(a))) = S(3a + 2) = 6a + 5 = Syr(S^2(a))$ $\square$

Corollary 2.3. *If a is of rank $n > 1$, $Syr(a)$ is of rank $n-1$, and $Syr(S(a)) = S(Syr(a))$*

Note 2.4. *The Syracuse action over an odd number is equivalent to adding 1 to it, then the half of the result, then -1. How many times one can add an half to an odd number $+1$ directly depends on the length of the immediate even branch of the binary tree that is to its right.*

Let us take Mersenne numbers for example, that are defined as $2^n - 1$. One can transform them consecutively in this way a number of time equal to their rank-1, indeed, 31, which is written 11111 is of rank 5, because $32 = 2^5$ so if one repeats the action "add to the number+1 the half of itself" which is equivalent to a multiplication by $\frac{3}{2}$ this will yield an even result ***exactly four consecutive times****.*

Thus, any ascending Collatz orbit concerns only numbers a of rank 2 or more, and is defined by

$(a + 1) \cdot (\frac{3}{2})^{rank-1} - 1$

because the rank is strictly indexed on the length of the next even branch of the binary tree on the right of the number, defining how many consecutive times the action $\cdot\frac{3}{2}$ will yield an even number.

Lemma 2.5. *Let a be an odd number of rank 1 that is vertical even, then $3a$ is successal, and $9a$ is vertical even.*

Let a be an odd number of rank 1 that is vertical odd, then $3a$ is successal, and $9a$ is vertical odd.

Proof. If a is vertical even it can be written $8k+1\ \forall k$

$3a = 24k+3$ and this number admits an S^{-1} that is

$12k+1$, which is an odd number, therefore $3a$ is Successal

$9a = 72k+9$ and this number admits a V^{-1} that is

$18k+2$, an even number indeed.

Now if a is vertical odd, it can be written $8k+5\ \forall k$

$3a = 24k+15$ and $9a = 72k+45$, $3a$ admits an S^{-1} and $9a$ admits a V^{-1}, respectively

$12k+7$ and $18k+11$ and they are both odd. □

Theorem 2.6. *(regular quaternary equivalence)*
Let a be a number that is vertical even, then $(a) \equiv S(a)$ and $S^k(a) \equiv S^{k+1}(a)$ for any even k. Let a be a number that is vertical odd, then $S(a) \equiv S^2(a)$ and $S^k(a) \equiv S^{k+1}(a)$ for any odd k.

*We will call these relations merging alternate pairs of odd branches **"regular quaternary equivalences"** or **qe**.*

Proof. If a is vertical even then it can be written as $Gp)$ where p is necessarily vertical (odd or even)

so by lemma 2.5 we have that $3p$ is successal and by lemma 2.2 we have $Syr(S(a)) = G(p)$ so it is necessarily vertical odd (since $3d$ is successal) so $Syr(a) = V^{-1}(Syr(S(a))$ and therefore $a \equiv S(a)$

This behavior we can now generalize to the n, because if a is vertical even and of pivot p, then the lemmas we used also provide that $Syr^n(S^n(a)) = G(3^n \cdot p)$ and therefore $Syr^n(S^n(a))$ will be vertical even for any even n because $3^n \cdot p$ will be vertical something (even or odd, depending on what the pivot was) for any even n.

Now if a is vertical odd it can be written $G(p)$ and p is necessarily successal because $G \circ S = V$.

Thus $3p$ is vertical (even or odd) and therefore $Syr(S(a)) = G(3p)$ is vertical even. □

This **qe theorem** is a more elaborate, and now very useful building block of our demonstration, because it allows to place a relation of equivalence between every other pair of any odd branch of the binary tree, up to infinity. It is also based on the characteristics of increasing phases of the Collatz orbits: any number of rank n is finitely turned into a number that has a vertical, either odd or even.

We should now interest ourselves with the decreasing phases of the Syracuse orbits [1]. These decreasing phases only concern the glacis.

Theorem 2.7. *(glacial decreasing)*
Let a be a vertical even number with a glaciality of $\begin{bmatrix} n \\ b \end{bmatrix}$ *where n is even, then* $a \equiv 3^{\frac{n}{2}}(b)$
Let a be a vertical even number with a glaciality of $\begin{bmatrix} m \\ b \end{bmatrix}$ *where m is odd, then* $a \equiv S(3^{\frac{m+1}{2}}(b))$

Proof. if a is of glaciality $\begin{bmatrix} n \\ b \end{bmatrix}$

then by definition $a = 2^{n+2}b + 1$.

Then $3 \cdot a + 1 = 3(2^{n+2}b + 1) + 1) = 2^{n+2} \cdot (3b) + 4$.

As this expression can be divided by 2 no more than twice, we have

$Syr(a) = 2^n 3b + 1$, therefore the glaciality of $Syr(a)$ is

$$\begin{bmatrix} n-2 \\ 3b \end{bmatrix}$$

Note that if $n = 2$ then $V^-(Syr(a)) = V^-(2^2 \cdot (3a) + 1) = 2^2 \cdot \frac{1}{4} \cdot (3b) = 3b$ which is of course an odd number. Therefore $Syr(a)$ is vertical odd and $V^-(Syr(a)) = 3b$ thus we have proven that $a \equiv 3b$.

If $n = 1$ then $a = 2^3 \cdot b + 1$ so $3(a + 1) = 2^3 \cdot 3b + 4$ therefore $Syr(a) = S(3b)$ and thus $a \equiv S(3b)$.

From this we can generalise the progression of glacis numbers. Let b be any odd number, thus defining a glacis bottom. All **"Variety S"** numbers of its glacis are written $V(b \cdot 2^{2k-1})$ or $S(b \cdot 2^{2k}) = 2^{2k+1} \cdot b + 1$ and all **"Variety V"** numbers of its glacis are written $V(b \cdot 4^k)$ or equivalently $S(b \cdot 2^{2k+1}) = 4^{k+1} \cdot b + 1$. Any glacis number g of order $2k$ or $2k - 1$ may be thus reduced to a glaciality of 0 or -1 by the following transformation: $(g - 1) \cdot \left(\frac{3}{4}\right)^k + 1$ therefore we do have indeed that,

[1] remember that we are still only considering odd numbers when we write "decreasing phases", and still defining $Syr(a)$ as "the next odd in the orbit of a"

- for Variety S numbers : $2^{2k+1} \cdot b \cdot \left(\frac{3}{4}\right)^k + 1 = 2b \cdot 3^k + 1 = S(b \cdot 3^k)$

- for Variety V numbers : $4 \cdot 4^k \cdot b \cdot \left(\frac{3}{4}\right)^k + 1 = 4b \cdot 3^k + 1 = V(b \cdot 3^k)$

As the obtaining of these equalities will fit any odd number b, we have that any glacis number of glaciality $2k$ will be finitely mapped to $4b \cdot 3^k + 1 = V(b \cdot 3^k)$ and that any glacis number of order $2k - 1$ will be mapped to $2b \cdot 3^k + 1 = S(b \cdot 3^k)$. In plain English:**any glacis number (and by the *qe*, also its successor) merges either directly with a power of three of its bottom or with the successor of it.**. This result too may seem anecdotal, but it is actually irreplaceable to describe the underlying generator of the "Collatz Seaweed" □

3 General description of the Syracuse dynamic

The **qe theorem** allows to place an infinite amount of equivalences along the binary tree over $\mathbb{N}$.

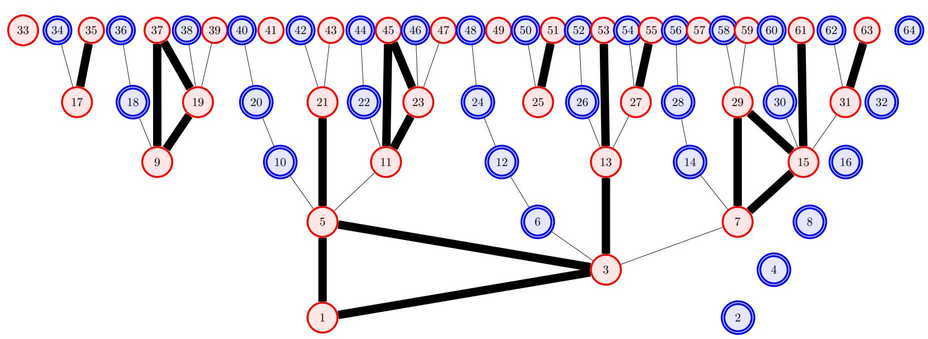

Figure 2. A representation of the intersection of the binary and the quaternary trees over $\mathbb{N}$: as only the quaternary operation $\cdot 4 + 1$ really matters to the representation, we have just warped the binary tree so that $V(a)$ is indeed *Vertical* to a. We have also not connected the rank 1 numbers to the even ones of which they are the successors so as to make glacis more easily visible. The bold black lines indicate the **qe** and vertical equivalences, and so whenever numbers are joined by a connective series of those lines, their Collatz orbits merge. Connecting all of those equivalences together completely solves the Syracuse problem, and as we will see in the next section, this endeavour requires the introduction of a third dimension: that of the ternary tree over $\mathbb{N}$.

3.1 Ascending phases

The orbit of an odd number can only increase if it is of rank 2 or more. Odd branch numbers of rank $n > 2$ ascend with the progression $+1 \cdot \frac{3}{2}^{n-1} - 1$ and this allows to compress their orbit to the next glacis.

More particularly though, if a pair is not connected by the **qe** equivalence, **it is because the rank 1 reduction of its smallest number the Collatz action is vertical odd**. When we have a pair $\{a; S(a)\}$ where a is vertical odd, a is mapped to $Syr(V^{-1}(a))$ and $S(a)$ is mapped to the glacis of bottom $Syr(V^{-1}(a))$ and this phenomenon, by which a pair of numbers that were related on the binary tree become separated we call an **Avalanche**. Avalanches account for absolutely all of the chaoticity of the Collatz orbits, and whoever can perfectly predict their occurrence and consequences shall have cracked *all* of the Collatz problem, beyond eve demonstrating that all natural numbers finitely converge to one.

An example of Avalanche can be understood on **figure 2** by observing the pair $\{3; 7\}$ of which the first Collatz image is $\{5; 11\}$ where 5 indeed is vertical odd. The avalanche is that 5 is mapped to the image of 1, which is 1, and 11 is mapped to 17, which is in the glacis of bottom 1. This happens to all such pairs, which we will call **"buds"**.

The alacrious reader will not fail to notice that 17 is precisely a glacis number of Variety V, and this is not happening by chance: if a vertical odd number a is the finite vertical of an even number, then $S(a)$ will be mapped as a variety V in the next glacis, and proportionally as high as a was vertical, and if the bottom of its verticality is odd, then $S(a)$ will be mapped as a variety S.

We have also noted that any power of 9 of a vertical number is either of the series "vertical odd" or the series "vertical even" and these two are parallel: one cannot obtain a vertical odd number by applying any power of nine to a vertical even one. Thus, the destiny of branches the determinant of which falls within the "vertical odd" or "vertical even" series is quite different, and this level of precision can also help understand the Collatz dynamic better. Still, we shall not use this interesting property in this work, even though it is more significant than a simple curiosity.

3.2 Descending phases

Any odd number can only decrease in *Syr* if it is of rank 1, and then it intersects either the Successor or the Vertical of a power of 3 of the bottom of its glacis. Note that in so doing, it always encounters the consequences of another **qe** on the branch it meets: either the power of three of the bottom of the glacis can be proven to merge with its successor, either it is vertical odd or it is its predecessor that is merging with it.

Here too, powers of 3 of the bottom of the glacis, just as being the case with odd branch determinants had useful implications to elaborate more advanced theorems, can either be of the "vertical odd" or "vertical even" sequences.

Numbers of even glaciality n decrease following the dynamic $-1 \cdot (\frac{3}{4})^{\frac{n}{2}} + 1$ and those of odd glaciality m, $-1 \cdot (\frac{3}{4})^{\frac{m+1}{2}} + 1$.

Thus, **whenever one can prove that an odd number merges with its triple, one proves that it merges with its first number of even glaciality**

Also, **whenever one can prove that an odd number merges with the successor of its triple, one proves that it merges with its first number of odd glaciality.**

We cannot stress enough the importance of this pair of results.

4 Using the ternary tree to connect the quaternary equivalences

4.1 Definitions

In the previous sections we have mostly identified numbers by their position in the complete binary and quaternary trees over $\mathbb{N}$. Using statements of the kind "a is vertical even" is a typical example of crossing binary and quaternary properties to identify specific characteristics of a number. We will now expand this methodology by adding the ternary tree, which elements we will identify with the following definitions:

Definition 4.1. *Ternary, Ternarity*
A number b is ***ternary*** *or of* ***type B****, if it can be divided by* 3*. Its* ***ternarity*** *is the total number of times it can be divided by* 3*, to which we will add the information of the non-ternary number resulting from this finite operation, thus the full ternarity of any ternary number that can be written* $3^n \cdot x$ *where* x *is non ternary is* $\begin{bmatrix} n \\ x \end{bmatrix}$ *For all intent and purpose, when we will refer to just the "ternarity" of a B type number (as opposed to "full ternarity") we will just be meaning* n *alone, namely, the. number of times it is dividable by* 3*.*

Definition 4.2. *1-ternary, 1-ternarity*
A number c *is* ***1-ternary*** *or of* ***type C****, if its base 3 representation ends with digit 1. The number of times one can remove a consecutive end digit 1 (we call this operation* C^{-1}*) is its* ***1-ternarity****, to which we add the information of the number resulting from it. Thus, a number* c *that can be written* $x\underbrace{1\dots1}_{n}$ *in base 3 has a 1-ternarity of* $\begin{bmatrix} n \\ x \end{bmatrix}$

Definition 4.3. *2-ternary, 2-ternarity*
A number a *is* ***2-ternary*** *or of* ***type A****, if its base 3 representation ends with digit 2.*

*The number of consecutive times one can remove an end digit 2 (we call this operation A^{-1}) is its **2-ternarity**, to which we add the number resulting from it. Thus, a number a that can be written $x\underbrace{2\dots2}_{n}$ in base 3 has a 2-ternarity of* $\begin{bmatrix} n \\ x \end{bmatrix}$

Thus, as a memorandum of the types of odd numbers in the full ternary tree over $2\mathbb{N}+1$ in the word "ABC", B is dividable by 3, A+1 is and C-1 is, thus B is in the middle, A on the left and C on the right.

Definition 4.4. *"up", "down"*

*A number is called **"up"** if its **qe** makes it merge with its successor. If $B \equiv S(B)$ we call it a **Bup** and respectively for A and C, **Aup** and **Cup**. If $B \equiv S(B)$, as we necessarily have that $S(B)$ is of type C, we will call this C "down" or $C down$. If a number is vertical odd, it is "down", if it is vertical even, it is "up".*

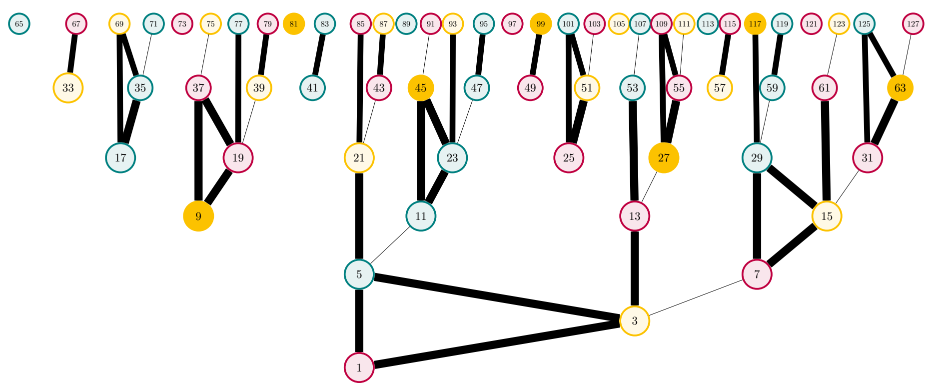

Figure 3. All odd numbers from 2^0 to 2^7. Type A are circled in teal, B in gold and C in purple. Numbers of a ternarity of 2 or more (numbers that can be divided by 9, that is) are also colored in gold. To explain again the previous definitions: 27 is a *Bup* and 63 is a *Bdown* for example. Any glacis number of glaciality above 0 is "up": 17 is an *Aup* for example, and 19 a *Cdown*. Though exotic, **these names are absolutely essential to the results we obtain**, and their very use is a pure result of our methodology: the "up" and "down" properties come from the study of the intersections of the binary and quaternary trees, and the A,B,C ones, from the ternary.

5 The Collatz quiver

5.1 Fundamental rules

From the theorems of the previous section we may now define the formal rules generating a particular **traversal quiver** over the binary tree defined by actions G (left arrow) and S (right arrow) over $2\mathbb{N}+1$. These rules apply anywhere on the set of odd numbers, where they define an endomorphism.

- **Rule One**
  $\forall x$ odd, $V(x) \equiv (x)$
- **Rule Two**
  $\forall x, k$ odd, $S^k V(x) \equiv S^{k+1} V(x)$ and $\forall x, k$ even, $S^k V(x) \equiv S^{k+1} V(x)$
  (this is the **qe**)
- **Rule Three**
  $\forall n, y \in \mathbb{N}^2$, $\forall x$ odd non B, $3^n x \equiv y$
  $\Rightarrow \bigwedge_{i=1}^{n} (V(4^i 3^{n-i} x)) \wedge S(V(4^i 3^{n-i} x)) \equiv y$
- **Rule Four**
  $\forall n, y \in \mathbb{N}^2$, $\forall x$ odd non B, $S(3^n x) \equiv y$
  $\Rightarrow \bigwedge_{i=1}^{n} (S(4^i 3^{n-i} x) \wedge S^2(4^i 3^{n-i} x)) \equiv y$
- **Rule Five**
  $\forall n \in \mathbb{N}$, $\forall y \in \mathbb{N}$, $\forall x$ odd non B where $3^n x$ is of rank 1, $a \equiv y$, $a = G(3^n x)$
  $\Rightarrow \bigwedge_{i=0}^{n} (S^i(G(3^{n-i} x)) \wedge S^{i+1}(G(3^{n-i} x))) \equiv y$

Applying these rules from number 3 generates what we will call the **Collatz 3-quiver**, which number of branches grow exponentially, in a density which we will explore in further details. As the five rules do apply anywhere on the set of odd numbers, one may develop an **n-quiver** from any odd number n as well. **importantly, demonstrating that any n-quivers has at least one common node (i.e., number) than the 3-quiver solves the Collatz problem.**

5.2 Examples

Let us proceed with numbers 3 and 21

$3 = 3 \cdot 1$ therefore by **rule 3** we have $3 \equiv 17$

by **rules 1 and 2** we now have $3 \equiv 69 = 3 \cdot 23$ and $3 \equiv 141 = 3 \cdot 47$

also, $17 = G(3^2 \cdot 1)$ so by **rule 5** and **rule 1** we have $3 \equiv 45 \equiv 93 \equiv 7 \equiv 15$ and by **rule 4** we have $7 \equiv 9$ so starting by 3 only we have already proven the convergence of all odd numbers up to 31 with the exception of 27 and 31, but we already have by **rules 4 and 5** that $31 \equiv 27$ and we will also show that the Golden Automaton finitely proves that $15 \equiv 31$

We have $21 = V^2(1) = 3 \cdot 7$ so **rules 1, 2 and 3** give us that 113, 453, 227 and 909 converge while **rules 5 and 2** further demonstrate the convergence of 75 and $151 = S(75)$ and we keep going...

So any number than can be written $3^n a$ or $S(3^n c)$ solves a glacis B number and any number that can be written $3^n c$ or $S(3^n a)$ proves a glacis A number.

In turn, any glacis A number, which can always be written $G(3^n x)$ proves a cup or a bup, and on the way, strictly more than 2n type B numbers as well

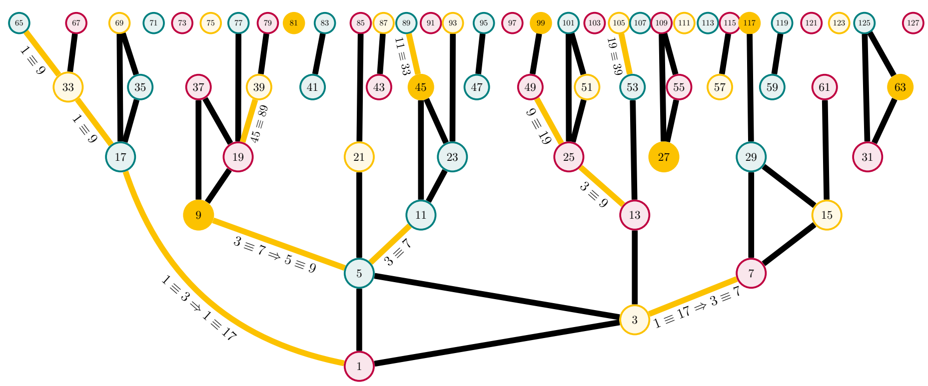

Figure 4. The golden branches on this figure are only a *subset* (and precisely: the *Bup* series after exploiting 3) of the equivalences the 3-quiver proves to reach them: assuming number 3 already grafted to the equivalence network (since we start from $1 \equiv 3 \equiv 5$), they are the ones generated by only considering the *Bups*, and not the (many) more opportunities offered by connecting *Bdowns* like 21, 45 or 69.

5.3 Emerging properties of the Collatz quivers

Though the development of the 3-quiver is branching, as each B type number that is vertical even provides with both an A type and a B type number to keep applying respectively rules 5 and 3, we may follow only the pathway of type A glacis numbers to define a single non-branching series of arrows, forming a single infinite branch of the 3-quiver which we may call ”pure A”. The latter, if computed from number 15, leads straight to 31, solving a great deal of other numbers on the way:

Definition 5.1. *A A_g or ”glacial A” is a type A number that is vertical even.*

$15 \equiv 81$	glacial type B (the only one, as we follow a ”pure A” branch from here)
$81 \equiv 1025$	A_g
$1025 \equiv 303$	branch Bup
$303 \equiv 809$	A_g
$809 \equiv 159$	branch Bup
$159 \equiv 425$	A_g
$425 \equiv 283$	branch Cup
$283 \equiv 377$	A_g
$377 \equiv 111$	branch Bup
$111 \equiv 593$	A_g
$593 \equiv 175$	branch Cup
$175 \equiv 233$	A_g
$233 \equiv 103$	branch Cup
$103 \equiv 137$	A_g
$137 \equiv 91$	branch Cup
$91 \equiv 161$	A_g
$161 \equiv 31$	branch Cup
$31 \equiv 41$	A_g
$41 \equiv 27$	branch Bup

Again, it is in no way a problem, but rather a powerful property of Collatz n-quivers that this particular segment of the infinite series of arrows (each being a *quiver branch*) already cover 19 steps (and actually more than that) because each of them is branching into other solutions. Simply put, any type A number in a glacis solves a Cup or a Bup (and on its way, many more numbers), which in turn brings either directly another glacial type A or a glacial type B giving itself another glacial type B and a glacial type A.

We may follow another interesting sequence to show that in the same way that Mersenne number 15 solves Mersenne number 31, Mersenne number 7 solves Mersenne number 127, this time we will follow a B branch up to the A_g of 127 which we know can be written $G(3^6)$ because 127 is the Mersenne of rank 7. By rule 4 we have the first equivalence

$7 \equiv 9$	Glacial Bup
$9 \equiv 25 \equiv 49$	both Glacial type C

So by Rule 2 we also have

$25 \equiv 51$

and rule 3 still gives us:

$51 \equiv 273$ Glacial Bup

so Rule 3 again gives us:

$273 \equiv 1457 = G(729) \equiv 127$

The 3-quiver thus generates an infinity of branches, each infinite collections of arrows between odd numbers. Importantly, the rules of the Collatz n-quivers ensures that its branches wrap themselves diagonally around the binary tree over $2\mathbb{N}+1$. **The 3-quiver is a diagonal *branching traversal* of the complete binary tree over $2\mathbb{N}+1$ with a non-uniform branching factor of at least 3 at every node**. Applying its founding rules in the regular order over $2\mathbb{N}+1$ is also an extremelu powerful algorithm to solve problems in the Collatz Conjecture.

Indeed, note that the only problems on the way of solving the Collatz conjecture have always been **Cups** and **Bups**, because by induction, whoever could prove that any Cup or Bup has a lower number than itself in its orbit would trivially solves Collatz just out of rules 1 and 2. Furthermore, any Cup or Bup finitely maps to a Glacial A, therefore any glacial A that is grafted by the 3-quiver solves a Cup (like 31) or a Bup (like 27). Besides, we have proven that the Collatz dynamic makes it so that any Cup or Bup is finitely mapped to another Cup or Bup, because any glacial A is finitely mapped to either a B or a C type number by glacial decreasing, and any Cup or Bup of course, is finitely mapped to a glacial A number.

The outline of our proof will consist of demonstrating that the 3-quiver solves too many glacial A numbers - both densely and diagonally - so too many Cups and Bups - to allow for any non-converging trajectory to exist over $2\mathbb{N}+1$. Simply put, the 3-quiver branches too much and covers too much, and since any supposedly non-converging trajectory will generate its own n-quiver, our strategy consists of demonstrating that the footprint of any such n-quiver, for any supposedly non-converging Cup or Bup, grows too much not to intersect the 3-quiver.

In turn the set of Cups and Bups can be measured very easily in terms of its density over $2\mathbb{N}+1$:

7 is the first Cup, 7+12 is a Cdown, 7+24 is a Cup, 7+36 is a Cup, 7+48 is a Cdown, etc. If we study the series of C types of rank >1 (which is a section of the binary tree over $2\mathbb{N}+1$) it gives

7 (U) - 19 (D) - 31 (U) - 43 (U) - 55 (D) - 67 (D) - 79 (D) - 91 (U) - 103 (U) - 115 (D) - 127 (D) - 139 (U) - 151 (D) - 163 (D) - 175 (U) - 187 (U) - 199 (U) - 211 (D) - 223 (D) ...

This is an **infinite braid** made of infinitely many strands of the C of rank>1, with each same-rank strand of the braid following the rule $U \to D \to U$. As the same goes with the number crystal of B type of rank>1, simply put, for any 7 consecutive numbers in $2\mathbb{N}+1$, a maximum of 3 are Cups and Bups and a minimum of 1 is, and as the binary tree develops, less than 16% of odd numbers are up, half of them being of a rank of 2 (like 27) meaning they are finitely mapped to another "up" of higher rank (e.g. 31 in the case of 27), which we may prove here.

We can indeed generalize the case of 27 in a theorem, to rigorously explain why all Cups and Bups of rank 2 could be eliminated already, i.e that any up of rank 2 is either converging to 1 straight away, or at least finitely mapped to a higher rank:

Theorem 5.1. *Second order rank theorem*

let $y = S(V(x))$ *where* x *is of rank* n*, then* y *is finitely mapped to* $(y+5)\cdot(\frac{9}{8})^{\lfloor\frac{n}{2}} - 5$

Proof. Though rather esoteric, this little theorem is actually very useful, both in methodology and consequences. We have already established as a founding theorem that any number x of rank n is finitely mapped to $(x+1)\cdot(\frac{3}{2})^{n-1} - 1$ which is of rank 1. In this theorem we can now define the *second order rank* of a "bud" - ie. a number that can be written $S(V(x))$ where x is odd - as the rank of x itself, and we can demonstrate that any such bud will finitely decrease in second order rank and grow by increments of $\frac{9}{8}$ by being mapped to numbers of glaciality 1, then back to branches in a rank of 2.

For 27 we have indeed $(27+5)\cdot\frac{9}{8} - 5 = 31$

For 59 we have $(59+5)\cdot\frac{9}{8} - 5 = 67$

The general demonstration consists of observing that any number $y = S(V(x))$ where x is of rank 2 or more is mapped to $G^2(Syr(x))$ which can only be a number of glaciality 1, so it is mapped in turn to $S(3\cdot S^{-1}(Syr(x)))$ thus giving our general formula

For a geometric intuition of the process, it could be summed up as "go to the closest number dividable by 8 on the right of the bud" (hence the +5) then compute one increasing Syr (branch process) and one decreasing Syr (glacis process), and that's the $\frac{9}{8}$, then give the +5 back. It is essentially the same process as computing Syr^n of a rank n odd number, only second order, in that it now concerns the bud right above said number.

□

As we have already noted any bud that is vertical even is strictly decreasing, that is, if a number can be written $S(V^2(e))$ where e is even, then it maps to a type A number of glaciality 2, and therefore maps straight to $3\cdot Syr(V(e))$ which is a strictly decreasing process.

So altogether, this eliminates all the Cups and Bups of rank 2 as problematic in the establishment of a final resolution of the Syracuse problem, so we now have at most 8% of odd numbers to worry about even before the 3-quiver is taken into account, and actually, as we will see, much less than that.

5.4 Density and Diagonality of the 3-Quiver

Rules 3, 4 and 5 ensure that the development of the 3-quiver over the Binary tree is *diagonal*, which is absolutely essential. Finishing a solution of the Collatz conjecture consists of demonstrating it is also *dense* enough not to allow any trajectory it does not capture. This would demonstrate at the same time that no cycles can exist in the Collatz dynamic but of course, that simply all natural numbers finitely converge to 1 in this discrete dynamical system.

As many chaoticians have observed since Lorenz, and chief among them Stuart Kauffman, chaos can actually remain quite "boxed" in its space phase, and this is exactly the case of the $3n + 1$ dynamical system, which in spite of its deterministic chaoticity exhibits rather precise attractors within $2\mathbb{N} + 1$. However, if the Collatz orbits converge to certain key numbers that can be beautifully visualised in the stems of the so-called "Collatz Seaweed", the 3-Quiver generates strictly more branches than there are real numbers and has therefore more branches than the complete binary tree over $2\mathbb{N} + 1$. In this current, and penultimate part of our demonstration, we will lay the foundations of this collision analysis to prove that any n-quiver intersects the 3-quiver in finite time.

The way the 3-quiver works, any type B and type C number proven to converges also provide either a Glacis A or a glacis B number, which in turn provide respectively a series of new B numbers and a branch Cup or Bup (in the case of a glacis A). Yet, glacis A numbers can be of any glacial order a priori, though a first step in "boxing" more the diversity of Collatz orbits into what we will call "Determinants", would consist of, for example, demonstrating that any Collatz orbit has its own footprint among the set of triple C numbers (which, by rule 3, is strictly equivalent to the set of glacial A numbers of order 2). To this end, we prove the following simple lemma.

Lemma 5.2. *Banana Split Lemma*

Let g_2 and g_1 be glacis numbers of ranks 2 and 1 respectively then either

- $g_2 \equiv g_1$ *("Vanilla")*
- $g_2 \equiv \frac{g_2 - 9}{8}$ *and* $g_1 \equiv (g_1 \cdot 8) + 9$ *("Banana")*

  We will call the action $8x + 9$ "Banup" and the action $\frac{x-9}{8}$ "Bandown"

Proof. By the **glacial decreasing theorem** (3.7) we have $g_2 \equiv 3b$ where b is the bottom of the glacis. Then b is either up or down. If it is up, $b \equiv S(b) \to g_2 \equiv g_1$. If it is down, $S(3b) \equiv S^2(3b)\ equiv V(S^2(3b))$ the latter being of type C, therefore $g_1 \equiv Banup(g_1)$. If $3b$ is down we also have either $3b = V(a)$ where a is of type A, or $3b = S(c)$ where c is of type C, either way it implies that $g_2 \equiv Bandown(g_2)$ □

It is also easy to demonstrate that any banana starting from a type A (and either going up or down) gives a type C number, and that any Banana from a type B gives another type B.

This little lemma, though anecdotal in appearance is actually very useful because it will guarantee that any trajectory merges with triple C or triple A numbers, which in turn is a an extremely important property of the Collatz Dynamic. Indeed, it implies other useful consequences:

Lemma 5.3.
- *Let* $3{\cdot}a = V(x)$ *where* x *is odd and* a *of type A, then* $Banup(3{\cdot}a) = 3{\cdot}c$ *where* c *is of type C and the pivot of the branch of* x *is dividable by* 9.

  In plain English: any type A branch with a pivot dividable by 9, that is, a branch made of numbers that have been increasing at least twice under the Syr operation has Triple A numbers above itself and triple C as any g_2 *numbers in its glacis*

- *Let* $V(a)$ *where* a *is of type A be dividable by* 9*, then* $Banup(V(a))$ *is dividable by* 9 *but neither is* $V(S(a))$ *or* $Banup(V(S(a)))$*. If V(S(a)) is a triple B (dividable by 9) then V(S(a)) is a triple C and Banup(V(S(a)) is a triple A. If a is of rank* 2 *or more, then* $V(S(a) \equiv Banup(V(S(a)))$

  In plain English: let a Cup or Bup increase just once under Syracuse, then one will find triple As and triple Cs it merges with right above its forward trajectory in the binary tree over $2\mathbb{N}+1$

Proof.
- It is trivial to demonstrate that by the Collatz Dynamic, the action $\cdot 3$ on any A branch gives a series of numbers that are vertical, except for the first number of the branch, for which the triple must be successal; also, since we have tripled an A branch (which pivot is therefore necessarily of type B) the pivot of the second branch is dividable by 9. Now does $Banup(3a)$ give a $3a$ number? Yes it does $\frac{24a+9}{3} = 8a + 3$ which is of type A.
- What the second case meant in plain English is that if we apply the action $\cdot 3$ on a C-B branch now, we obtain a series of B numbers that are vertical of an A branch (again, except for the root of the CB branch). Then either these verticals are 3B numbers, and are dividable by 9, or they are triple Cs, and that is the reason one will find triple Cs above any A branch of which the pivot is dividable by 3 only once.

□

This allows us to further narrow the set of problems by formally defining it as, at most, a certain proper subset of the set of triple Cs that are Vertical odd, or a certain proper subset of the set of triple As that are Vertical odd. 69 is the first triple A that is vertical odd, and the next one, 141 which it happens to be equivalent to by rules 1 and 2, is found at a step of +72 over $\mathbb{N}$, so that we now have reduced the set of problems to below 2,8% of the odd numbers, evenly distributed of course over their set, and this without having even used the 3-quiver yet. We may affirm that the set of actual problems is a proper subset of those less than 2,8% of odd numbers specifically **because any Cup or Bup of rank n will, in its trajectory, intercept as many of those as its orbit inflates**.

Furthermore, one must also note that any Triple A or Triple C also constitutes the starting point of the grafting of infinitely many new odd numbers, so whenever a supposedly diverging Cup or Bup expands its footprint in the set of problems (which we later call a **Collatz Determinant**) it does so exponentially. If its orbit inflates, it will intersect other Triple As and Cs but also other Cups and Bups forward, but each of the Triple

As and Cs it intersects will themselves solve another infinite series of Cups and Bups, backward. Simply put: **diverging implies dense branching** which we will further demonstrate in the final subsection.

From a Ramsey-theoretical perspective, it would be fascinating to define not some Collatz determinant as we have still rather broadly done here, but an absolute **Critical Set** defined as the smallest set of odd numbers - if it exists - whose solving solves Collatz. Such an endeavour is nevertheless not required at all to finish the proof, as the 3-quiver simply solves way too many problems, and the problems' footprint simply inflate way too much if we just assume they are problems (namely, that they dodge the 3-quiver's own footprint), that a collision between the 3-quiver and any supposedly non-connected n-quiver is inevitable anyway.

For any glacis B number we have an increasing and a decreasing pathway, which we saw in the example $15 \rightarrow 31$. Any Glacis B number finitely points to a pair of A and B glacis numbers, and, if it is dividable by 9 or more, to as many Bdown numbers as it is dividable by 3 minus one time. We now have that only Bups and Cups of rank strictly above 2 are problems in Syracuse, and that each time they grow they intersect twice as many triple As as their rank-1, plus one. Any number which orbit is supposed to escape the Golden Quiver will form an exponentially growing Silver Quiver within the space of Triple As. All we have to do now is to demonstrate that the Golden Rules starting from 3 grow too many solutions to avoid any possible Silver Quiver. This can be done in many ways but we would like to take the opportunity of this demonstration to expand upon the concept of Romanesco Calculus. Thus, although it could be demonstrated in a more straightforward manner, we will develop here a slightly more technical approach to build upon the more general interest of the methods and tools we have conceived in this article. We do not want to miss the opportunity to apply Romanesco algebra to a second order, that is, to now map all the triple A numbers to a 2-4 Romanesco and analyse their trajectory from this new referential. Again, this was not necessary in the first place, but the future potential of this sort of analysis warrants breaking ground for it in this very article already.

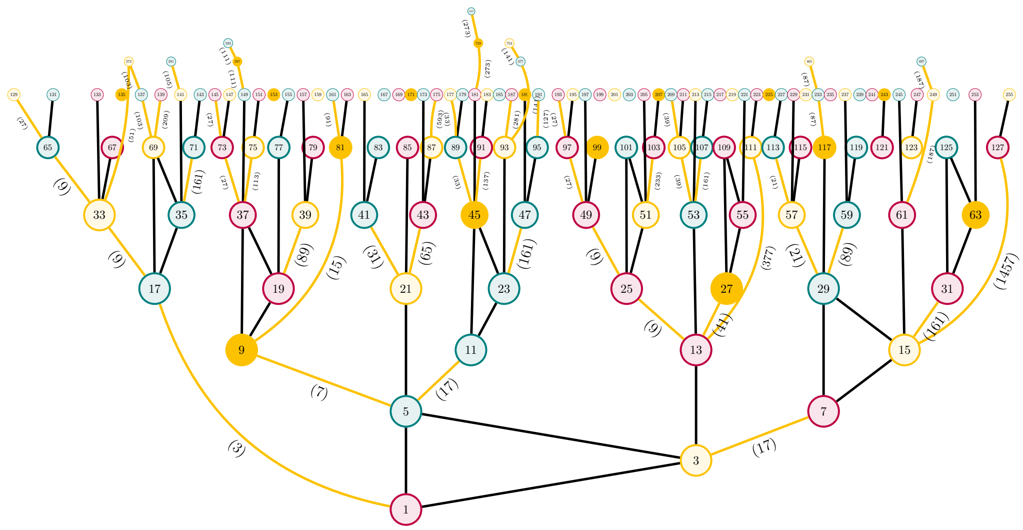

Figure 5. Note that the development of the 3-quiver is dual to the "Collatz Seaweed", yet it is not constructed by the brute-force computing of each odd number. This more complete description of the 3-quiver, grafting the **qe** together as it progresses from $1 \equiv 3 \equiv 5$, goes all the way to grafting 127 and 31. Each branch indicates which number generates it, for example 9 is grafted to 5 because 7 is grafted to 3, and 9 being a *Bup* it grafts 33 and 65.

5.5 Quiver analysis of Collatz determinants

In this final part of our proof, we demonstrate that the diagonality and density of the 3-quiver interdicts the existence of any n-quiver defined by the property of not intersecting it.

Paramount to this demonstration is the analysis of the consequences of the 5 rules, and in particular rules 3, 4 and 5 implying, among others, that

- any triple A solves a glacial Bup
- any Bup solves a glacial A and a glacial Bup
- any glacial A solves a branch Cup or a branch Bup

but also, still by those same rules 3, 4 and 5, we may focus on the case of glacial Bup 81, which we have demonstrated finitely solves Mersenne number 31. Since 81 is of ternarity 4, the rules imply it solves $2 \cdot (4-2) = 4$ triple Cs on its way to mapping to the single glacial A-B pair it solves, and any triple C solves a glacial A, which solves a branch Cup or Bup, and on its way, other triple As and triple Cs. The important part of this final step to solving the Collatz problem is the analysis of the geometric consequences of action $\cdot 3$ on branches and glacis.

- let $G_x = \{g_1 ... g_n\}$ be a glacis of bottom x, then $3 \cdot G_x = S(G_{3x})$ to which is appended $3 \cdot g_1 = S(V(3x))$
- let $B_x = \{x, S(x) ... S^n(x)\}$ be a branch of root x, then $3 \cdot B_x = V(Syr(B_x))$ to which is appended $G(Syr(S(x))$

These consequences of the five rules ensure, among others, that any supposedly escaping orbit would also leave an exponentially growing quiver of triple Cs and triple As as a footprint. For example, the mere orbit of 7 also provides, among others, Triple A 483, (by banana between $241 = S^{-1}(483)$ and $29 = V(7)$, which had to be because of the rank of 7) but also triple A 753 and Triple C 93 by banana over $23 = S(Syr(7))$ and finally triple As 69 and 141 (over $17 = Syr^2(7)$) and triple C 1137 (by banana, again inevitable from the rank of 7 , from 1137 to 141)

A further step, though unnecessary *per se* to solve the Collatz problem, consists of mapping all the triple A numbers that can be written as the vertical of an A type number and establish thiss Collatz Determinant within this space phase that any orbit has to intersect, also noting that any triple A provides a quiver of new solutions.

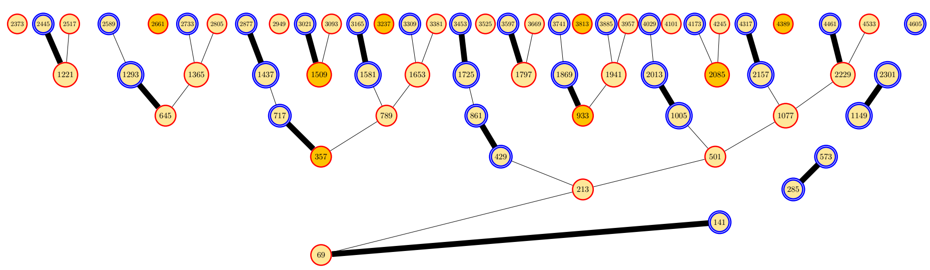

Figure 6. Representation of a Collatz Determinant mapped onto a binary tree over $\mathbb{N}$, made of the Triple A numbers that can be written $V(a)$ where a is of type A. Each number is associated with a single integer, circled in blue if even, and red if odd. For any integer n, its associated Triple A is equal to $72n - 3$. The solid black lines plot the consequences of rules 1 and 2 onto this representation. Only a subset of the Triple A numbers that are "vertical odd" in this referential (i.e. that admit, in this representation, a V^{-1} that is itself a Triple A of odd integer n ; they are displayed in darker gold and they are +576 apart from each other in nominal value e.g. $357 + 576 = 933$) actually needs to be solved to solve the Collatz problem and that the 3-quiver solves an infinity of Triple A numbers in a density that is sufficient to interdict the existence of any other independent n-quiver finishes the proof.

Simply put; the process generating problems (i.e. Cups and Bups of rank 2 or more) cannot win against the process generating solutions (glacial As), by virtue of its diagonality and the arities (plural, because they are not the same at every node) of its nodes, which are anywhere strictly above 2, and in fact, progressive ones indexed on the ternarity of the B nodes : taking the case of 81 as an example, we have this number of ternarity 4 mapping to 513 (glacial B) and 1025 (glacial A) but also, on its way, to triple A 771, triple B 1539, triple C 579, triple C 1155, triple C 867 and triple C 435, giving a final arity of 8 to this node, which depends only on the ternarity of 81. The 3-quiver solves problems faster than they can appear, and each presumed problem grows a footprint that cannot escape it. Another approach to finish the proof consists of noting that there is a bijection from the set of successal Cups and Bups to the set of glacial As, yet a proper surjection from the same set to the set of triple C numbers (each representing another glacial A of order 2) and that while only a proper subset of numbers of ternarity 1 needs to be solved, numbers of ternarity strictly above 1 solve exponentially more of those (e.g. case 81).

If any n-quiver is finitely intercepted by the 3-quiver then not only cannot there be any kind of cycle in the 3n+1 dynamic, but every number also finitely converges to 1.

5.6 Romanesco Algebra

We now, in this final step, formalize the results of the quiver analysis of the Collatz orbits into a precise algebraic system. This system of unary algebras over $2\mathbb{N}+1$ we here call a "Romanesco Algebra"

Definition 5.2. *Romanesco Algebra*

The unary algebra $\{2\mathbb{N}+1; G, S, V, \cdot 3\}$ *is called a "Romanesco Algebra (2;3;4) over odd numbers".*

Examples

- The complete Binary Tree with the extra V operation over $2\mathbb{N}+1$ (**Figure 1**) forms a "Romanesco (2;4) over odd numbers"
- **Figure 3**, identifying odd numbers by their ternarity while mapping them in a binary tree where V is visible, is a representation of Romanesco (2;3;4).

Romanesco arithmetic may be seen as an epistemological extension of modular arithmetic, hence our use of the symbol $\equiv$ in this article. Romanesco arithmetic involves words taken in the alphabet $\{G; S; V; 3\}$, which we will call in their order of application, just like in turtle graphics. For example VGS3 means $3 \cdot S \circ G \circ V$

Rules 3, 4 and 5 may now be reformulated as such, without loss of generality

- **Rule 3**
  Let b be of type B, then $b \equiv VGS3^{-1}$ from b
  We will call this action R_b

- **Rule 4**
  Let c be of type C, then $c \equiv GS3^{-1}$ from c
  We will call this action R_c

- **Rule 5**
  Let a be of type A, then $a \equiv V3^{-1}S^{-1}$ from a
  We will call this action R_a

We have already proven that the Collatz dynamic produces a one-to-one correspondence between all the Bups and Cups of rank 2 or more, and we know by the second order rank theorem that any of those is also finitely mapped to a Bup or Cup of rank 3 or more. So solving the A_g of order 2 in a glacis which bottom is of rank 4 or more would solve Collatz, but such a delicate Ramsey-theoretical pursuit is not needed to demonstrate that the 3-quiver finitely intersects any n-quiver.

Theorem 5.4. *The 3-quiver finitely intersects any n-quiver.*

Proof. A_g numbers are 12 points apart on $2\mathbb{N}+1$ (24 in nominal value, e.g. 17 to 41) and any Bup or Cup they represent is smaller than them since action R_a is strictly decreasing so up to the n^{th} A_g there are $2n$ (Bups + Cups) of rank 2 or more and half of them are equivalent to these A_g (e.g. between 17 and 41 Bup 27 is equivalent to A_g 41, which is equivalent by glacial decreasing to Cup 31)

between any two consecutive A_g in $2\mathbb{N}+1$ there are

- 8 non-A numbers
- 1 of them at most is mapped to the second A_g
- 3 at most are ups of rank 2 or more

Besides,

- Let b be of type B, there are $\frac{2b}{3}$ numbers of type A_g that are smaller than $V^2(b)$
- Let c be of type C, there are $\frac{S(c)}{3}$ numbers of type A_g that are smaller than $V^2(c)$
- Let 3c be a type B where c is of type C, there are $\frac{S(c)}{3}$ numbers of type A_g up to $R_b(3c)$ included
- Let 3a be a type B where a is of type A, there are $\frac{G(a)}{3}$ numbers of type A_g smaller than $R_b(3a)$

These consequences of Rules 1 to 5, which again do apply anywhere on $2\mathbb{N}+1$ allow but one conclusive property, of which we saw isolated, though significant examples when we calculated that 15 is finitely demonstrating the convergence of 31 and 7 finitely demonstrating the convergence of 127, both demonstrating an exponential amount of other non-trivial convergences on the way, that is :

let numbers $2^n + 1$ ***to*** $2^{n+1} - 1$ ***be proven to converge, then applying Rules 1 to 5 on them and their successor by those rules finitely demonstrate the convergence of*** $2^{n+1} + 1$ ***to*** $2^{n+2} - 1$. □

6 Conclusion

Pál Erdős famously said of the Collatz conjecture that mathematics may not be ready for this kind of problem. What we have attempted here has been to provide a simple theory of unary algebras, and especially of complete n-ary trees over $2\mathbb{N}+1$, which we have called "Romanesco Algebra". Unary algebras of this kind could be particularly fecund to attack other diophantine problems and we have in no way intended Romanesco algebra to be an ad hoc "fire and forget" theory made to measure for the Collatz conjecture. Rather, we predict its founding methodology and principles be used to approach other unsolved problems in Number theory and discrete dynamic systems.

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