On the Syracuse conjecture over the binary tree

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Abstract

We investigate recurrent behaviors of the $\frac{1}{2}(3x+1)$ action ("Syr") over the binary tree. The original composition of otherwise simple algebraic behaviors (eg: if $a = 4b+1$, then $Syr(a) = 4(Syr(b))$) allows us to identify the existence of a relation merging orbits over alternate pairs of any odd branch of the tree. We prove that it is possible to compress any Syracuse orbits to critical positions in the tree and give new hints as to how to predict those positions. This allows us to establish a novel research program for the resolution of the Collatz conjecture, of which we also introduce original, and simpler conjectures. This article proves that for any even number e, the orbits of $Vert(e) := 4e+1$ and $S(Vert(e)) := 2(4e+1)+1$ merge, but also that for any k that is even, $S^k(Vert(e))$ and $S^{k+1}(Vert(e))$ can be proven to merge. For any odd number o, $S(Vert(o))$ and $S^2(Vert(o))$ also merge, and so do any $S^k(Vert(o))$ and $S^{k+1}(Vert(o))$ for any odd number k. Another significant result is that for any odd number o, either $8o+1$ and $16o+1$ merge or $8o+1$ merges with $16(Vert(o))+1$ and $16o+1$ merges with $2o-1$. The main result of this paper is that for any odd number o, proving that the orbits of $4o+1$ and $2(4o+1)+1$ merge will also prove the Collatz conjecture. That such orbits could be systematically proven to merge we call the Golden Gate Conjecture and outline some aspects of a research program attacking this conjecture.

1 Introduction

In this paper we study the $3x+1$ problem, also known as the Collatz problem, the Syracuse conjecture, Kakutani's problem and several other names.

The $3x+1$ problem concerns the iteration of the following function

$$T(x) = \begin{cases} \frac{3x+1}{2}, & \text{if } x \equiv 1 (mod 2); \\ \frac{x}{2}, & \text{if } x \equiv 0 (mod 2). \end{cases}$$

The Syracuse conjecture asserts, that for all $x \geq 1$ exist a number $k \in \mathbb{N}$ such that $T^k(x) = 1$. Iterations of the T function are however known to produce complicated - albeit strictly deterministic - orbits when recurrently applied to natural numbers. As these orbits are still very poorly understood, the Syracuse conjecture is currently unsolved, and considered a textbook example of wild deterministic chaos,

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so much so that Paul Erdos famously declared "mathematics may not be ready for such problems".[1] (p.57). A good start to attack Syracuse then, would consist of making the problem - and the tools - ready for further enquiry. In this paper we attempt to identify, predict and compose some of the behaviors of the Syracuse orbits in the binary tree. The interest of this approach is twofold. Firstly, successful predictions could more thoroughly open the problem to artificial intelligence. Secondly, it could expose otherwise unknown algebraic vulnerabilities, as well as a more general strategy to attack problems in discrete mathematics. As we will prove, our attack of the Syracuse problem already yields the significant result that demonstrating that, for any odd integer o, the orbits of $4o+1$ and $2(4o+1)+1$ merge proves the conjecture.

We begin by representing all numbers in the form of the following tree, in which all numbers $x \in \mathbb{N}$ are vertices. The lowest level of the tree consists of numbers 1 and 2 with number 0 (not represented here) at level 0. Every level $n > 1$ of this tree consists of all numbers from $2^{n-1} + 1$ up to 2^n.

Any vertex a is connected with two vertices $2a$ and $2a + 1$. In the figure we will show the following connections. If $a > 1$ and $a \neq 2^k$ the vertex $2a$ will be connected with $2^2 a$. And for any $a > 1$ the vertex $2a + 1$ will be connected with $2 \cdot (2a + 1)$ and $2 \cdot (2a + 1) + 1$.

We will call a number x *red* if it is odd, and if x is even we will call it *blue*.

The Figure 1 thus shows the first five levels of the tree, with red numbers marked with one circle and blue numbers with two circles.

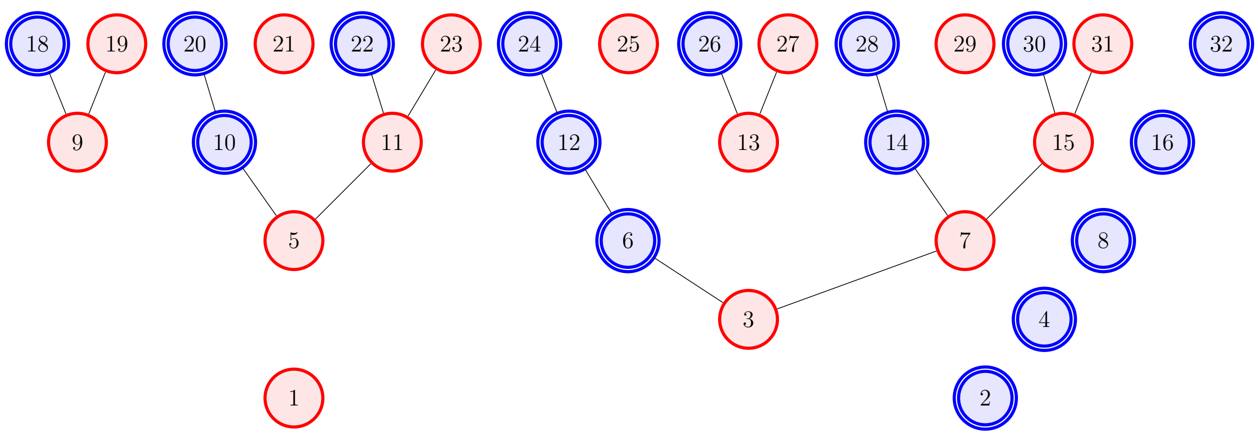

Figure 1. Binary tree

We now wish to find regular behaviors of the Syracuse orbits of any numbers within this tree.

2 Decreasing of the rank

Let us consider an action $\rho(x) = \frac{(x-1)}{2}$. The *rank* of odd number x, denoted by $rank(x)$, is defined to be the number of times we will need apply ρ to the element x until we get an even number. The rank is defined as positive non-zero only for odd numbers ; even numbers are of rank zero by definition. A number $x = 2^{n+1} \cdot t + (2^n - 1)$ has a rank n. (*This is a general formula for a n-rank number.*)

We call an element y an *image* of x if $y = Syr(x)$, this action is defined by the following expression

$$Syr(x) = \frac{(3x+1)}{2} = \frac{3}{2}(x+1) - 1.$$

Denote $\underbrace{Syr(\ldots Syr}_{n}(x)) = Syr^n(x)$.

We will call the sequences generated by $3x+1$ and $/2$ the *orbit* of element x and denote this sequence by $Orb(x)$.

Theorem 1. *Let $rank(x) = n$, where $n > 1$. Then $rank(Syr(x)) = n - 1$.*

Proof. The number $x = 2^{n+1} \cdot t + (2^n - 1)$ is odd and $rank(x) = n$. We first find a successor of x

$$Syr(x) = \frac{1}{2}(3x+1) = \frac{1}{2}\left(3 \cdot \left(2^{n+1} \cdot t + (2^n - 1)\right) + 1\right) = \frac{1}{2}\left(2^{n+1} \cdot 3 \cdot t + 2^n \cdot 3 - 3 + 1\right) =$$

$$= \frac{1}{2}\left(2^{n+1} \cdot 3 \cdot t + 2^n \cdot 3 - 2\right) = 2^n \cdot 3 \cdot t + 2^{(n-1)} \cdot 3 - 1.$$

Further we perform a few algebraic transformations

$$Syr(x) = 2^n \cdot (3 \cdot t) + 2^{(n-1)} \cdot (2+1) - 1 = 2^n \cdot (3 \cdot t) + 2^{(n)} + 2^{(n-1)} - 1 = 2^n \cdot (3t+1) + (2^{(n-1)} - 1).$$

By the formula of a n-rank number we find that $rank(Syr(x)) = n - 1$. This finishes the proof. □

3 Further definitions and statements

For the next two definitions let the number x have a $rank(x) = 1$.

Definition 1. *We call a number x vertical blue if $\frac{1}{4}(x-1)$ is even.*

Definition 2. *We call a number x vertical red if $\frac{1}{4}(x-1)$ is odd.*

The set of integers is closed under the action $(4x + 1)$ but not closed under the action $\frac{1}{4}(x-1)$. Let $Vert^+(x) = Vert(x) = 4x + 1$ and $Vert^-(x) = \frac{1}{4}(x-1)$. Note that $Vert^+(Vert^-(x)) = x$ and $Vert^-(Vert^+(x)) = x$.

Definition 3. *The orbits of two numbers are said to merge if they have at least one common number.*

Definition 4. *Let x be odd and $y = Vert^+(x)$ then x is v-related to y.*

Proposition 1. *Let x and y be v-related. Then the orbits of x and y merge.*

Proof. Since x is v-related to y, we have $y = Vert^+(x) = 4x + 1$. Then $\frac{1}{4}Syr(y) = \frac{1}{8}(3(4x+1)+1) = \frac{1}{8}(12x+4) = \frac{1}{2}(3x+1) = Syr(x)$. Since $(\frac{1}{4}Syr(y)) \in Orb(y)$ and $\frac{1}{4}Syr(y) = Syr(x)$ the orbits of x and y merge. □

Denote $S(a) = 2 \cdot a + 1$ and $\underbrace{S(\ldots S}_{n}(a)) = S^n(a) = 2^n a + 2^n - 1$, we call $S(a)$ a *successor* of a.

Definition 5. *Let a be an odd number of rank 1. The infinite set of numbers $\{a, S(a), S(S(a)), \ldots\}$ is called a red branch of root a.*

Definition 6. *Let x be an odd number. If there is such a number y that $Syr(y) = x$, then x is called an odd number of type A and a remainder of dividing x by 3 is 2. If there is such a number y that $Syr(y) = 2x$, then x is called an odd number of type C, and a remainder of dividing x by 3 is 1. If x is divisible by 3 it is not of type A or type C and we call it of type B. Also note that if x is a type B number then $4x+1$ is of type C and therefore there is such a y that $Syr(y) = 2(Vert^+(x))$. In any case then, the orbits of x and y merge.*

Proposition 2. *For any odd number x, there are infinitely many Syracuse orbits either leading to it or to $Syr(x)$.*

Proof. Let x be an odd number. If x is divisible by 3, then it is of *type B* and there is a number y so that $Syr(y)$ is the double of $Vert^+(x)$. If $x+1$ is divisible by 3 then it is of *type A* and there is a number y so that $Syr(y) = x$. If $x+2$ is divisible by 3 then it is of *type C* and there is a number y so that $Syr(y) = 2x$.

As for any y so defined there are infinitely many $Vert^+(y)$ numbers merging to its orbit, and for any x so defined, also infinitely many positive numbers k defining $Vert^k(x)$ with each of them having at least one z of which the orbit will merge with either itself, its double, or the double of its vertical, there are indeed infinitely many orbits leading to the forward orbit of any odd number. □

It is therefore possible, for any odd number x, to define a certain Syr^- function.

Definition 7. *We call a r-related to* $S(a)$ *if* $Syr^{(rank(a)-1)}(a)$ *is vertical blue.*

Theorem 2. *Let a be vertical blue, then* $Syr(S(a))$ *is v-related to* $\frac{1}{2}Syr(a)$.

Proof. Let a be vertical blue, then $b = Vert^-(a)$ is even. Hence $b = 2k$, then $a = Vert^+(b) = 8k+1$. Then

$$\frac{1}{2}Syr(a) = \frac{1}{2}\Big(\frac{1}{2}\Big(3(8k+1)+1\Big)\Big) = \frac{1}{4}(24k+4) = 6k+1, \text{ and}$$

$$Syr(S(a)) = Syr(2(8k+1)+1) = Syr(16k+3) = \frac{1}{2}\Big(3(16k+3)+1\Big) = 24k+5.$$

Since $(6k+1)$ is odd and $Vert^+(\frac{1}{2}Syr(a)) = Vert^+(6k+1) = 4(6k+1)+1 = 24k+5$, number $Syr(S(a))$ is v-related to $\frac{1}{2}Syr(a)$.

□

Let us remark that $Syr(S(a)) = 24k+5$ is vertical red because $Vert^-(24k+5) = 6k+1$ is odd. From the proof we also have that $Vert^-(Syr(S(a))) = \frac{1}{2}Syr(a)$.

Proposition 3. *For any number* $a \in \mathbb{N}$ *and natural number* $n \leq rank(a)$ *we have the following identity*

$$S(Syr^n(a)) = Syr^n(S(a)).$$

Proof. We will consider only $n \leq rank(a)$ because for any greater n the $Syr^n(a)$ is not defined in a sufficiently monotonous manner (see "Avalanche" later in this article), as $Syr^{rank(a)}(a)$ will be an even number.

We will use a mathematical induction to prove the statement.

Let $n = 1$ then

$$S(Syr(a)) = S(\frac{1}{2}(3a+1)) = 2 \cdot \frac{1}{2}(3a+1)+1 = 3a+2 = \frac{1}{2}(3(2a+1)+1) = Syr(S(a)).$$

Then let us assume that for any number less than k the identity $S(Syr^k(a)) = Syr^k(S(a))$ holds. Then we will show that it holds for $k+1$

$$S(Syr^{k+1}(a)) = S(Syr^k(Syr(a))) = Syr^k(S(Syr(a))) = Syr^k(Syr(S(a))) = Syr^{k+1}(S(a)).$$

Hence $S(Syr^n(a)) = Syr^n(S(a))$ holds for any number $a \in \mathbb{N}$ and natural number $n \leq rank(a)$.

□

Proposition 4. *Let a be vertical blue, then* $Syr(S(a))$ *is vertical red.*

Let a be vertical red, then $Syr(S(a))$ *is vertical blue*

Proof. Let a be vertical blue with $Vert^{+}(b) = a$ then

$$Vert^{-}(Syr(S(a))) = Vert^{-}\left(\frac{1}{2}\Big(3(2a+1)+1\Big)\right) = Vert^{-}\left(\frac{1}{2}\Big(6a+3+1\Big)\right) = Vert^{-}(3a+2) =$$

$$= \frac{1}{4}\Big((3a+2)-1\Big) = \frac{1}{4}(3a+1) = \frac{1}{4}\Big(3(4b+1)+1\Big) = \frac{1}{4}(12b+4) = 3b+1.$$

Then if b is an even number, $3b+1$ has to be odd, and if b was an odd number, $3b+1$ would be even. □

Proposition 5. *Let a be r-related to $S(a)$. Then a and $S(a)$ merge.*

Proof. Let a be r-related to $S(a)$ then $Syr^{(rank(a)-1)}(a)$ is vertical blue by definition and $rank\Big(Syr^{(rank(a)-1)}(a)\Big) = 1$.

By Proposition 3

$$Syr\left(S\Big(Syr^{(rank(a)-1)}(a)\Big)\right) = Syr\left(Syr^{rank(a)-1}\Big(S(a)\Big)\right) = Syr^{rank(a)}(S(a)).$$

Since $Syr^{(rank(a)-1)}(a)$ is vertical blue, by Theorem 2 number $Syr(S(Syr^{(rank(a)-1)}(a)))$ is v-related to $\frac{1}{2}Syr(Syr^{(rank(a)-1)}(a))$.

Hence by the Proposition 1 numbers $Syr^{rank(a)}(S(a))$ and $\frac{1}{2}Syr^{rank(a)}(a)$ merge. Then we get that a and $S(a)$ merge.

□

Definition 8. *The rank of pair $\{a, S(a)\}$ is equal to $rank(a)$.*

Remark 1. *Note that Proposition 3 actually explains the alternative distribution of r-relations along red branches:*

(i) let the root a of an red branch be vertical red then for any k that is odd we have $S^k(a)$ and $S^{k+1}(a)$ r-related;

(ii) if for some odd k numbers $S^k(a)$ and $S^{k+1}(a)$ r-related, then the root a of an red branch is vertical red;

(iii) let the root a of an red branch be vertical blue then for any k that is even we have $S^k(a)$ and $S^{k+1}(a)$ r-related;

(iv) if for some even k numbers $S^k(a)$ and $S^{k+1}(a)$ r-related, then the root a of an red branch is vertical blue.

Proof. (i) Let a be the root of an red branch, and vertical red. Then $Vert^{-}(a) = 2k+1$ is an red number, then $\frac{1}{4}(a-1) = 2k+1$ and $a = 8k+5$. Then $S(a) = 16k+11$ and $S^2(a) = 32k+1$. The pair $(S(a), S^2(a)\ )$ is the first pair of even rank. To check if $S(a)$ is r-related to $S^2(a)$ we should check if $Vert^{-}(Syr^{rank(S(a))-1}(S(a)))$ is red or blue. We have that

$$Vert^{-}(Syr^{rank(S(a))-1}(S(a))) = Vert^{-}(Syr(16k+11)) = Vert^{-}(24k+17) = 6k+4.$$

Since $6k+4$ is an even number $Syr^{rank(S(a))-1}(S(a))$ is vertical blue and $S(a)$ is r-related to $S^2(a)$.

Let us then find the rank of pair $(S(a)^{2t-1}, S^{2t}(a))$

$$rank(\ (S^{2t-1}(a), S^{2t}(a))\ ) = rank(S^{2t-1}(a)) = 2t.$$

Then we can find that $S^{2t-1}(a) = 2^{2t-1}a + 2^{2t-1} - 1$ and $Syr^{2t-1}(S^{2t-1}(a)) = 3^{2t-1}a + 3^{2t-1} - 1$.

Hence

$$Vert^{-}(\ Syr^{2t-1}(S^{2t-1}(a))\ ) = \frac{1}{4}(3^{2t-1}a + 3^{2t-1} - 2) = \Big| a = 8k+5 \Big| = \frac{1}{4}(3^{2t-1}(8k+5) + 3^{2t-1} - 2) =$$

$$= \frac{1}{4}(3^{2t-1}8k + 5\cdot 3^{2t-1} + 3^{2t-1} - 2) = \frac{1}{4}(3^{2t-1}8k + 6\cdot 3^{2t-1} - 2) = \frac{1}{4}(3^{2t-1}\cdot 2\cdot 4\cdot k + 2\cdot 3\cdot 3^{2t-1} - 2) =$$

$$= 3^{2t-1}\cdot 2\cdot k + \frac{1}{4}(2(3\cdot 3^{2t-1} - 1) = 3^{2t-1}\cdot 2\cdot k + 2\frac{1}{4}(3^{2t} - 1) = 3^{2t-1}\cdot 2\cdot k + \frac{1}{2}(3^{2t} - 1) =$$

$$= 3^{2t-1}\cdot 2\cdot k + \frac{1}{2}(3^t - 1)\cdot(3^t + 1).$$

Since $(3^t - 1)$ and $(3^t + 1)$ are even, $\frac{1}{2}\cdot(3^t - 1)\cdot(3^t + 1)$ is even and the whole expression $Vert^{-}(\ Syr^{2t-1}(S^{2t-1}(a))\ ) = 3^{2t-1}\cdot 2\cdot k + \frac{1}{2}\cdot(3^t - 1)\cdot(3^t + 1)$ is even. Hence $Syr^{2t-1}(S^{2t-1}(a))$ is vertical blue then $S^{2t-1}(a)$ is r-related to $S^{2t}(a)$.

(ii) Let $k = 2h - 1$ be an odd number and $S^k(a)$ is r-related to $S^{k+1}(a)$.

Hence $Syr^{(rank(S^k(a))-1)}(S^k(a))$ is vertical blue that is $Vert^{-}(Syr^{(rank(S^k(a))-1)}(S^k(a)))$ ia blue. We already have the next chain of equalities

$$Vert^{-}(Syr^{(rank(S^k(a))-1)}(S^k(a))) = Vert^{-}(Syr^{(rank(S^{2h-1}(a))-1)}(S^{2t-1}(a))) = Vert^{-}(\ Syr^{2h-1}(S^{2h-1}(a))\ ) =$$

$$= \frac{1}{4}(3^{2h-1}(8k+5) + 3^{2h-1} - 2).$$

Hence $\frac{1}{4}(3^{2h-1}a + 3^{2h-1} - 2)$ is even. Let us transform last expression

$$\frac{3^{2h-1}a - 3^{2h-1}}{4} + \frac{2\cdot 3^{2h-1}}{4} - \frac{2}{4} = 3^{2h-1}\cdot\frac{a-1}{4} + \frac{2}{4}(3^{2h-1} - 1) = 3^{2h-1}\cdot\frac{a-1}{4} + \frac{1}{2}(3\cdot(3^{2h-2} - 1) + 2) =$$

$$= 3^{2h-1}\cdot\frac{a-1}{4} + \frac{3\cdot(3^{h-1} - 1)(3^{h-1} + 1)}{2} + 1.$$

We know that $\frac{3\cdot(3^{h-1}-1)(3^{h-1}+1)}{2} + 1$ is odd. Since $3^{2h-1}\cdot\frac{a-1}{4} + \frac{3\cdot(3^{h-1}-1)(3^{h-1}+1)}{2} + 1$ is even we get that $3^{2h-1}\cdot\frac{a-1}{4}$ is odd. Hence $\frac{a-1}{4}$ is odd and a is vertical red.

The proof of the case (iii) and (iv) can be obtained analogously.

□

Definition 9. *Let a be an odd number. The Glacis of bottom a is an infinite set of odd numbers*

$$\{2a+1, 4a+1, 8a+1, \ldots, 2^n a + 1, \ldots\}.$$

Note that numbers $2^n + 1$, where $n \in \mathbb{N}$ are glacis numbers from the glacis of bottom 1.

Definition 10. *For each number $2^n a + 1$ in a glacis we define its glacis coordinates as the pair $(a; n-2)$, where the first coordinate is the unique glacis bottom and the second coordinate is called the glacis order of the element.*

By this definition the glacis coordinates of number $2a+1$ are $(a; -1)$, for number $4a+1$ they are $(a; 0)$ and number $8a+1$ has coordinates $(a; 1)$.

We will call glacis numbers $(a; 2k)$ with an even glacis order *Type Vert numbers* and the others shall be *Type Succ numbers.*

We will later explain the logic of this naming.

Proposition 6. *Let the glacis coordinates of* x *be* $(a; n)$*, where* $n > 0$*. Then the glacis coordinates of* $\frac{1}{2}Syr(x)$ *are* $(3a; n-2)$*.*

Proof. By the statement of the proposition $x = 2^{n+2}a + 1$. Then $\frac{1}{2}(Syr(x)) = \frac{1}{4}(3(2^{n+2}a+1)+1) = = 2^n \cdot (3a) + 1$. Hence glacis the coordinates of $\frac{1}{2}Syr(x)$ are $(3a; n-2)$. □

Note that if $n = 2$ the number $Vert^{-}(\frac{1}{2}Syr(x)) = Vert^{-}(2^2 \cdot (3a) + 1) = 2^2 \cdot \frac{1}{4} \cdot (3a) = 3a$ is odd. Therefore $\frac{1}{2}Syr(x)$ is vertical red.

If $n = 1$ and number y is of glacis order $m = 2$ in the same glacis then $\frac{1}{2}Syr(x) = S(Vert^{-}(\frac{1}{2}Syr(y)))$ This explains why we have called glacis numbers with an odd order "type Vert" and the ones with an even order "type Succ".

We may now generalize the formula to calculate the progression of glacis numbers. Let a be any odd number. All type Succ numbers of its glacis are written

$$Vert(a \cdot 2^{2k-1}) \text{ or } Succ(a \cdot 2^{2k}) = 2^{2k+1} \cdot a + 1$$

and all type Vert numbers are written

$$Vert(a \cdot 2^{2k}) \text{ or } Succ(a \cdot 2^{2k+1}) = 4^{2k} \cdot a + 1.$$

Any glacis number g of order $2k$ or $2k-1$ may be reduced to rank 0 or -1 by the following transformation

$$(g-1) \cdot \left(\frac{3}{4}\right)^k + 1$$

then on we have

(i) for type Succ numbers :

$$2^{2k+1} \cdot a \cdot \left(\frac{3}{4}\right)^k + 1 = 2a \cdot 3^k + 1;$$

(ii) for type Vert numbers :

$$4 \cdot 4^k \cdot a \cdot \left(\frac{3}{4}\right)^k + 1 = 4a \cdot 3^k + 1;$$

(iii)

$$S(a \cdot 3^k) = 2a \cdot 3^k + 1;$$

(iv)

$$Vert(a \cdot 3^k) = 4a \cdot 3^k + 1.$$

As these equalities will fit any number a, we have indeed that any glacis number of order $2k$ will be finitely mapped to $4a \cdot 3^k + 1 = Vert(a \cdot 3^k)$ and that any glacis number of order $2k-1$ will be mapped to $2a \cdot 3^k + 1 = S(a \cdot 3^k)$.

4 Main results

We shall now study the behavior of glacis numbers and establish the importance of studying the transformation of glacis under Syracuse.

Lemma 1. *Let $g1$ and $g2$ be glacis elements of glacis coordinates $(x;1)$ and $(x;2)$ respectively, and $3x$ be either vertical blue or r-related to $S(3x)$. Then $g1$ and $g2$ merge.*

Proof. Let elements $g1$ and $g2$ belong to the glacis of bottom x, then these elements are respectively of the form $g1 = 8x + 1$ and $g2 = 16x + 1$.

Let us study their orbits further. Firstly, we consider the elements of $Orb(g2)$. We have that

$$Syr(g2) = \frac{3}{2}(16x + 1 + 1) - 1 = 3(8x + 1) - 1 = 24x + 2$$

is an even number, then, the next number in $Orb(g2)$ is

$$\frac{1}{2}Syr(g2) = 12x + 1.$$

The number $\frac{1}{2}Syr(g2) = 12x + 1$ belongs to the glacis of bottom $3x$ and $3x$ is odd because x is odd. Then $(12x+1)$ is vertical red because $Vert^{-}(12x+1) = \frac{1}{4}((12x+1)-1) = 3x$ is an odd number. Hence $(12x + 1)$ and $3x$ are v-related therefore by Proposition 1 they merge. Hence $(16x + 1)$ and $3x$ merge.

Now we consider some consecutive elements of $Orb(g1)$.

$$\frac{1}{2}Syr(g1) = \frac{1}{2}(\frac{3}{2}(8x + 1 + 1) - 1) = 6x + 1.$$

The number $6x + 1$ also belongs to the glacis of bottom $3x$. Since $3x$ is vertical blue or r-related to $S(3x)$, by Remark 1 numbers $3x$ and $(6x + 1)$ merge. Thus $(8x + 1)$ and $3x$ merge.

Since $(16x + 1)$ and $(8x + 1)$ merge with the same number $3x$, these numbers also merge, which finishes the proof.

□

Definition 11. *When Lemma 1 holds for glacis numbers $g1$ and $g2$, namely, when $Syr(g1)$ and $Syr(g2)$ belong to a glacis of which the bottom is either vertical blue or r-related to its successor, we will call these numbers vanilla-related.*

Figure 2 shows how vanilla can be represented on the binary tree.

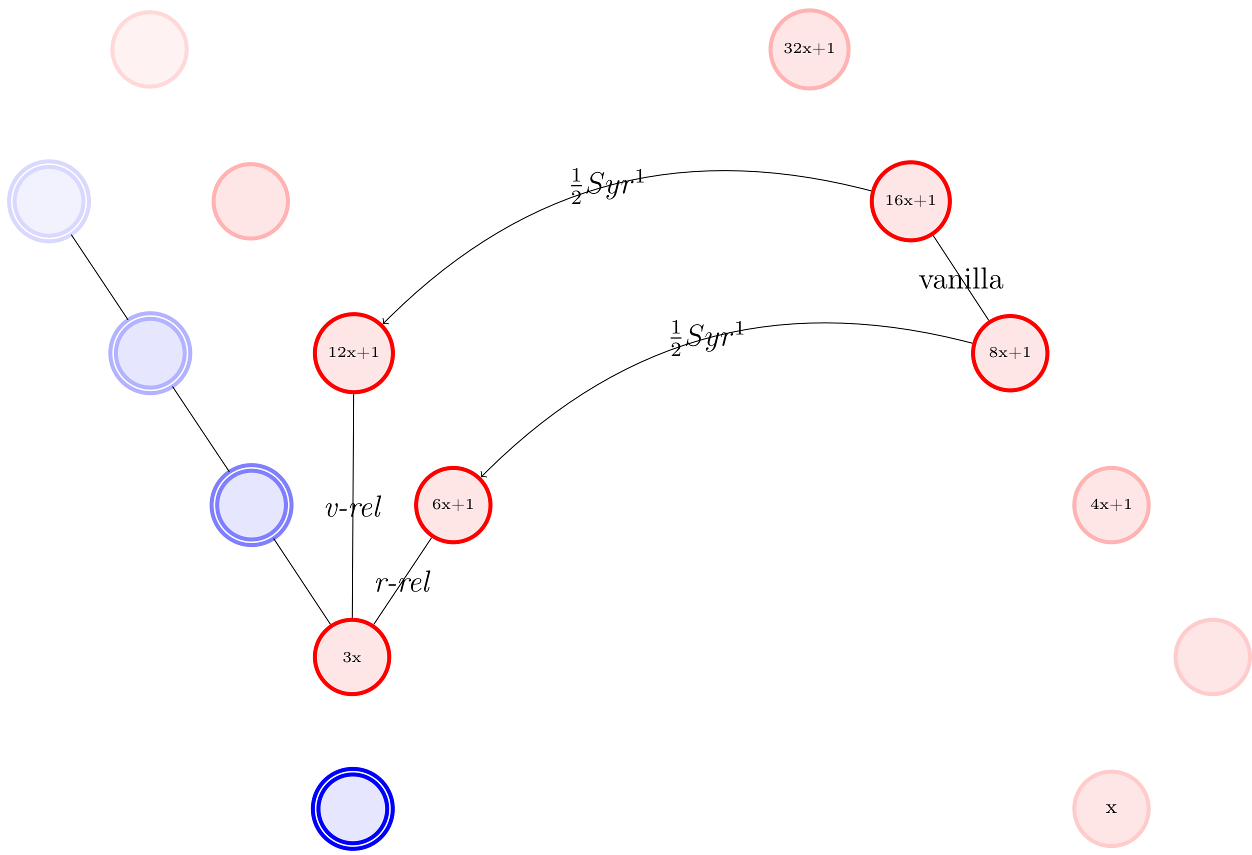

Figure 2. Vanilla

Let us define an action $Ban^+(a) = 8a + 9$ and $Ban^-(a) = \frac{1}{8}(a - 9)$. By default, just as we use the notation $Vert(a)$ for the exact $Vert^+(a)$ we will also use $Ban(a)$ for $Ban^+(a)$.

Lemma 2. *Let $g2$ have glacis coordinates $(x; 2)$, $g1$ have glacis coordinates $(x; 1)$*

Let us also define $g'2 = Ban^-(g2)$, $g'1 = Ban^+(g1)$, and $3x$ be vertical red or r-related to $\frac{1}{2}(3x - 1)$. Then $g1$ and $g'1$ merge, and so do $g2$ and $g'2$

Proof. Let us find how numbers $g1$, $g'1$, $g2$, $g'2$ depend from the glacis bottom x. We already have $g1 = 8x + 1$, $g2 = 16x + 1$. Also $g'1 = 8 \cdot (8x + 1) + 9 = 16(4x + 1) + 1 = 16 \cdot Vert(x) + 1$ and has glacis coordinates $(Vert^+(x); 2)$. And $g'2 = \frac{1}{8}(16x + 1 - 9) = 2x - 1$.

Since $\frac{1}{2}Syr(g2) = \frac{1}{2}(\frac{1}{2}(3(16x + 1) + 1)) = 4 \cdot 3x + 1 = Vert(3x)$, it follows that $g2$ and $Vert(3x)$ merge. Since by Proposition 1 numbers $Vert(3x)$ and $3x$ merge, we have that $g2$ and $3x$ merge. We have $\frac{1}{2}Syr(g'2) = \frac{1}{2}(\frac{1}{2}(3(2x - 1) + 1)) = \frac{1}{2}(3x - 1)$ from this $g'2$ and $\frac{1}{2}(3x - 1)$ merge.

Since $\frac{1}{2}Syr(g1) = \frac{1}{2}(\frac{1}{2}(3(8x + 1) + 1)) = 2 \cdot 3x + 1 = S(3x)$ we have that $g1$ and $S(3x)$ merge. The equality $\frac{1}{2}Syr(g'1) = \frac{1}{2}(\frac{1}{2}(3(64x + 17) + 1)) = \frac{1}{4}(192x + 52) = 48x + 13 = 4 \cdot (2 \cdot (2 \cdot 3x + 1) + 1) + 1 = Vert(S^2(3x))$ implies that $g'1$ and $S^2(3x)$ merge.

If $3x$ is vertical red then $Vert^-(3x)$ exists and $\frac{1}{2}Syr(g'2) = 2 \cdot \frac{1}{4}(3x - 1) = 2Vert^-(3x)$. Hence $\frac{1}{4}Syr(g'2) = Vert^-(3x)$ and $Vert^+(Vert^-(3x)) = 3x$ then numbers $g'2$ and $3x$ merge. Since $\frac{1}{2}Syr(g2) = Vert(3x)$ numbers $g2$ and $3x$ merge and we have that $g2$ and $g'2$ merge.

Let $3x$ be vertical red then by Remark 1 number $S(3x)$ is r-related to $S^2(3x)$. Then $S(3x)$ and $S^2(3x)$ merge. Hence $g'1$ and $g1$ merge.

Let $3x$ be r-related to $\frac{1}{2}(3x - 1)$ then by Proposition 5 numbers $3x$ and $\frac{1}{2}(3x - 1)$ merge. Hence $g2$ and $g'2$ merge. Since by the Remark 1 $S(3x)$ and $S^2(3x)$ merge it follows that $g'1$ and $g1$ merge

□

Definition 12. *If Lemma 2 holds for glacis numbers $g'1$, $g1$ and $g'2$ and $g2$ we will say that $g'1$, $g1$ are banana-related and that so are $g'2$ and $g2$*

Figure 3 shows how banana can be represented on the binary tree.

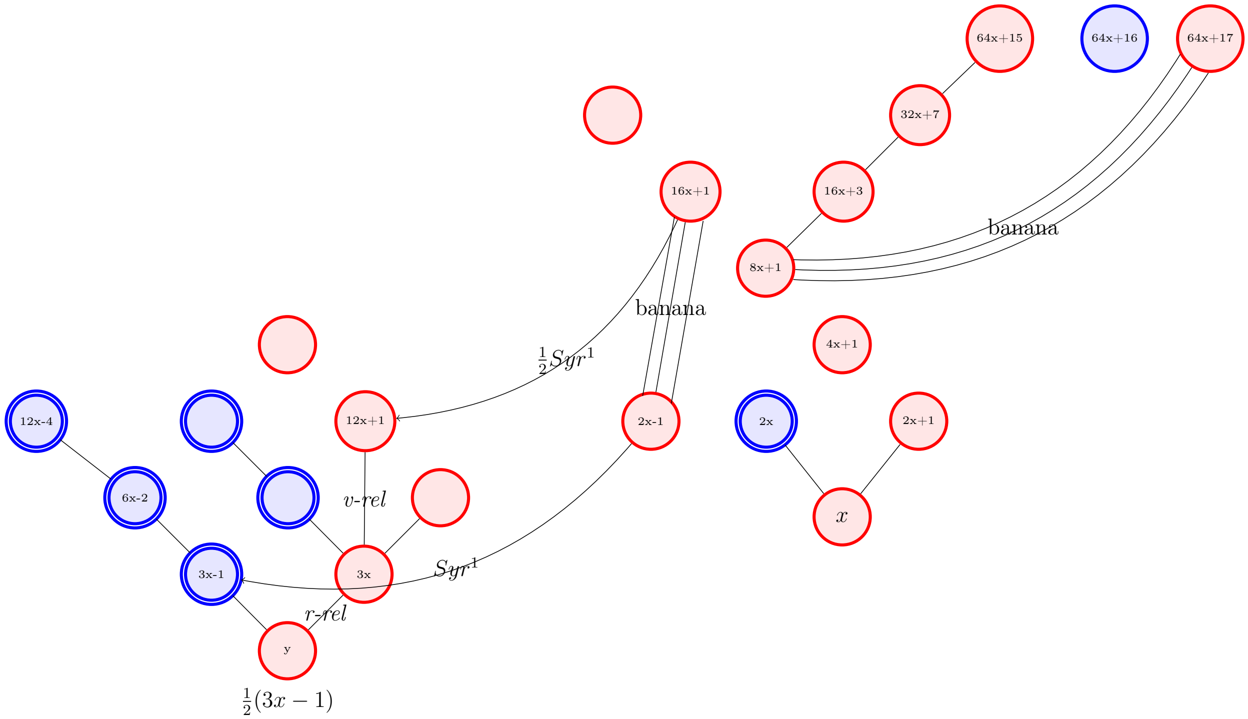

Figure 3. Banana

The algebraic phenomenon behind the Banana and Vanilla relations is the following:

for any two consecutive first glacis numbers $g1$ and $g2$ where $g1$ is of order 1 and $g2$ of order 2, and the glacis is of bottom a

$$\frac{1}{2}Syr(g2) = Vert(3a),$$
$$\frac{1}{2}Syr(g1) = Succ(3a).$$

Then only two cases are possible: either $3a$ is r-related to $S(3a)$ or it is not.

If $3a$ is r-related to $S(3a)$ then it means $g2$ and $g1$ merge.

If $3a$ is not r-related to $S(3a)$ it means $S(3a)$ is r-related to $S^2(3a)$ and that either $3a$ is vertical red or is r-related to $S^-(3a) = \frac{1}{2}(3a-1)$.

Now $3a$ is necessarily a *type B* number, therefore $Vert^-(3a)$, if it exists, is a *type A* number, so $3a-1 = 4 \cdot Vert^-(3a)$ has a number y such that $Syr(y) = 3a-1$ this number y is the number we have called $g'2$.

When $3a$ is r-related to $\frac{1}{2}(3a-1)$, this latter number is of *type C*, therefore there exists a number y such that $\frac{1}{2}Syr(y) = 3a-1$ which is number $g'2$ again. Since $S^2(3x)$ is r-related to $S(3x)$ and $3x$ is of *type B*, $S^2(3x)$ is of *type C*, therefore there is a number z such that $Syr(z) = 2 \cdot S^2(3x)$ and this number is the one we have called $g'1$.

Therefore we have the following :

Theorem 3. *(Banana-Split Theorem) For numbers $g2$ with glacis coordinates $(x;2)$ and $g1$ with glacis coordinates $(x;1)$, only one of the two following statements holds:*

(i) $g2$ and $g1$ merge;

(ii) $g2$ and $g'2 = Ban^-(g2) = \frac{1}{8}(g2-9)$ merge and $g1$ and $g'1 = Ban^+(g1) = 8 \cdot g1 + 9$ merge.

For the record, the author, having to find a name for the banana-relation on his handwritten notes, and being used to noting them as three black lines from one number to the other in the binary tree, with the three lines being curved, found it evoked a banana, hence the name.

Any Syracuse orbit is a correspondence mapping red branches to glacis and vice versa. By correspondence we mean that any Syracuse orbit is a sequence of odd numbers of which the rank is either greater than one, and they are therefore red branch numbers, or equal to one, in which case they are red branch roots and therefore glacis numbers.

Since any odd number of rank n greater than one will be reduced to an odd number of rank 1 in a finite number of Syr action, and that any odd number of rank 1 is in a glacis, any element of a red branch will finitely orbit to a glacis.

Glacis numbers, in turn, all finitely orbit to a red branch (get to a red branch in a finite number of steps), with only two possible patterns (ie. cases): either they are of $TypeVert$ and their glacis reduction to order 0 under a finite amount of $\frac{1}{2}Syr$ actions will be the $Vert^{+}$ of a power of three of the bottom of their initial glacis.

Or they are $TypeSucc$ and their glacis reduction to order -1 under a finite amount of $\frac{1}{2}Syr$ actions will be the Successor of a power of three of the bottom of their initial glacis.

Therefore, an important observation is that the forward orbit of any odd number

(i) If it is a red branch number of rank above 1, can only occupy red branches of *type A* until it reaches a glacis.

(ii) If it is a glacis number of order above 0, Can only occupy glacis of type B bottom (let us call it b) and therefore of *type C* glacis numbers until it reaches a red branch in either a $typeB$ number $(3^k \cdot b)$ or a *type C* number $S(3^k \cdot b))$.

Definition 13. *A bud is any pair $(a; S(a))$, where a is vertical red.*

Definition 14. *Let a be a number for which $Vert^{-}(a)$ is not an integer. If a is even, the cob of base a is the infinite set of buds*

$$\{(Vert^{+}(a); S(Vert^{+}(a)), (Vert^{+2}(a); S(Vert^{+2}a)), \ldots\}.$$

If a is odd, the cob of base a is the infinite set of buds

$$\{(a; S(a)), (Vert^{+}(a); S(Vert^{+}(a)), (Vert^{+2}(a); S(Vert^{+2}a)), \ldots\}$$

and it has the base a.

Since any number, either odd or even, has a $Vert^{+}$ number, there are indeed two kinds of cobs: those of blue base (*blue cob*), and those of red base (*red cob*).

Definition 15. *Let a be a vertical red number. The verticality of number a are the coordinates (b, m) such that b is the smallest number and m the highest number satisfying the equation $a = Vert^{m}(b)$.*

Definition 16. *The verticality of a bud $(Vert^{+n}(a); S(Vert^{+n}a))$ is equal to the verticality of its smallest element $Vert^{+n}(a)$.*

Proposition 7. *Let $(a, S(a))$ be a bud, and of verticality (b, m). (Nota Bene: that $(a, S(a))$ be defined as a bud implies that if b is even, m is greater than 1) Then if b is even, $Syr(S(a))$ will have the glacis coordinates $(\frac{1}{2}Syr(Vert(b)); 2(m-1))$ and will therefore be of type $Vert$. If b is odd, $Syr(S(a))$ will have the glacis coordinates $(\frac{1}{2}Syr(b); 2m-1)$ and will therefore be of type Succ.*

Therefore note that the image of any cob of base b under one Syr action is either, if the cob is of an even base, all the $typeVert$ glacis numbers of the glacis of bottom $\frac{1}{2}Syr(Vert(b))$, or if the cob is of odd base, all the typeSucc numbers of the glacis of bottom $Syr(b)$

Proof. If b is odd, then we know that $Syr(Vert^m(b)) = 4^m Syr(b)$ and $Syr(S(a)) = S(4^m Syr(b)) = 2(4^m Syr(b)+1)$ By definition of the glacis coordinates, $Syr(S(a))$ is therefore of coordinates $(\frac{1}{2}Syr(b); 2m-1)$.

If b is even, for any m greater than 1, $Syr(Vert^m(b)) = 4^{m-}Syr(Vert(b))$ And $Syr(S(a)) = S(4^{m-}Syr(Vert(b))) = 2(4^{m-1}Syr(Vert(b)) + 1)$ By definition of the glacis coordinates $Syr(S(a))$ is therefore of coordinates $(\frac{1}{2}Syr(Vert(b)); 2m-1)$ □

Proposition 8. *Let a be a glacis number of coordinates $(b, 2k+1)$ where k is greater or equal to 0*

Then defining $Tyr(a) := \frac{1}{2}Syr(a)$ $Tyr^{k+1}(a) = S(3^{k+1}Syr(b))$

if a is a glacis number of coordinates $(b, 2k)$ then $Tyr^{k+1}(a) = Vert(3^{k+1}Syr(b))$

Also note that in the same way that for x a number of rank $n > 0$ $Syr^n(a)$ may be calculated as $(x+1)\cdot(\frac{3}{2})^n - 1$.

For a glacis number of order $2k+1$ or $2k$ $Tyr^{k+1}(a)$ may be calculated as $(a-1)\cdot(\frac{3}{4})^{k+1} + 1$.

Proof. This is a re-stating of propositions that were already proven in the Banana-Split Theorem

□

Also, the latter two propositions implies that

Proposition 9. *Let $S(a)$ be the largest bud number of bud $(a, S(a))$, with vertical coordinates (b, m). Then if b is odd $Tyr^m(Syr(S(a))) = S(Syr(b)\cdot 3^m)$, if b is even then $Tyr^m(Syr(S(a))) = Vert(Syr(b)\cdot 3^m)$.*

Proof. This is also a re-stating of propositions that we proved earlier. □

Therefore, if we consider any red branch, and take a number a of this branch, with rank $n > 1$ that is r-related to $\frac{1}{2}(a-1)$ we know that $Syr^n(a)$ will be vertical red of a certain verticality, and therefore $Syr^n(a)$ and $S(Syr^n(a))$ will be forming a bud. If we could predict exactly the verticality (b, m) of the pair $(Syr^n(a), S(Syr^n(a)))$, we could know that either

$Tyr^m(Syr(S(Syr^n(a)))) = S(Syr(b)\cdot 3^m)$ or $Tyr^m(Syr(S(Syr^n(a)))) = Vert(Syr(b)\cdot 3^m)$

Meaning for any red branch number, we could predict the next red branch its orbit will intersect, namely the next time it will have a rank above 1, after it has been reduced to rank 1 for the first time by a finite number of Syr actions.

In a further work, we shall outline a method to achieve such a prediction.

Definition 17. *Let $(a, S(a))$ be a bud. Under one Syr, a is transformed into a power of 4 of the bottom of the glacis of which $Syr(S(a))$ will be an element. This critical separation of two adjacent red branch numbers of the binary tree into a non-adjacent pair composed of a glacis bottom and a glacis number we call an avalanche.*

The avalanche phenomenons accounts for a decisive part of the chaotic behavior of Syracuse orbits. In fact, the only two sources of the wild behavior of all Syracuse orbits are

Definition 18. *The consequence of the Syr^n on the vertical coordinates of buds, namely, for a bud $(a, S(a))$ of vertical coordinates (b, m) what will be the vertical coordinates of $(Syr^n(S^n(a)), Syr^n(S^{n+1}(a)))$? We call this problem the **B2G problem**, standing for "Branch to Glacis".*

Definition 19. *The consequence of the $3x$ action on the red branch coordinates of a number a, namely, given a number a that has a certain rank and of which the root of its branch has certain vertical coordinates, what will be the vertical coordinates and the rank of $3^n a$? We call this problem the G2B problem, standing for "Glacis to Branch".*

The next conjectures, which are in fact equivalent, would be helpful in searching for the solution of the Syracuse Problem.

Conjecture 1. *(Golden Gate Conjecture) Any bud is solvable, namely for any odd number a, the orbits of $Vert(a)$ and $S(Vert(a))$, either backward or forward, can be proven to have at least one common number.*

Conjecture 2. *(Golden Avalanche Conjecture) Any glacis number can be proven to merge with its glacis bottom.*

Proposition 10. *Proving the Golden Gate Conjecture proves the Collatz Conjecture*

Proof. Along any red branch, we know that if a is not r-related to $S(a)$ it means that $Syr^{rank(a)-1}(a)$ is vertical red and $Syr^{rank(a)-1}(S(a)) = S(Syr^{rank(a)-1}(a))$. Suppose we had a demonstration that for any odd number b, $Vert(b)$ and $S(Vert(b))$ can be proven to merge, it would imply that a and $S(a)$ will merge as well. Therefore, such a demonstration would mean that, besides the r-relation which we have exposed in this paper, there is a - evidently much more complex - w-relation between any two r-related pairs along any red branch. This in turn would demonstrate that all the elements of the binary tree will merge. □

For the record, the author of this article established the Golden Gate conjecture at the Lange Special Collection Reading Room of the University of California, San Francisco, with a view of the Golden Gate Bridge, a name altogether fitting for the definition of a "bridge" connecting two "red numbers" as they were colored in his personal notes. Obviously, proving the Golden Gate Conjecture will be no trivial work, but we may already make a few simple observations:

Proposition 11. *The solving of some buds implies the solving of some other buds*

Proof. Suppose bud $(13, 27)$ is solved. Number 13 belong to the orbit of number 45, and number 91 belong to the orbit of number 27. Indeed $\frac{1}{2}Syr(45) = 17$, $\frac{1}{2}Syr(17) = 13$ and $\frac{1}{2}Syr^2(27) = 31$, $\frac{1}{2}Syr(\frac{1}{2}Syr^5(31)) = 91$. And $(45, 91)$ is a bud. □

Proposition 12. *It is possible for a bud to solve itself by having each two orbits of its elements cross an composition of r-relation demonstrating they merge, and this without having to bruteforce compute the orbits of the two elements until they reach number 1.*

Proof. Let us consider the bud $(157, 315)$. and its forward orbit. $Syr(315) = 473$ and $Syr(157) = 236 = 4 \cdot 59$. Then $Syr(59) = 89$ and $\frac{1}{2}Syr(473) = 355 = S(177)$ and 177 is vertical blue therefore 355 and 177 are r-related. Numbers 89 and 177 are two consecutive glacis numbers with coordinates $(11, 1)$ and $(11, 2)$ respectively. If $3 \cdot 11 = 33$ was r-related to $S(33) = 67$, then 89 and 177 would be proven to merge by a vanilla relation. It so happens that 33 is vertical blue, and is therefore r-related to 67, implying a vanilla-relation between 89 and 177, because $\frac{1}{2}Syr(89) = 67$ and $\frac{1}{2}Syr(177) = 133 = Vert(33)$ Therefore 157 and 315 are proven to merge.

From a purely epistemological perspective, the chaotician that is experienced in theoretical biology may not fail to notice a certain intellectual similarity between the way bud $(157, 315)$ is solved and that in which new covalent bonds are made between different atoms by the active sites of enzymes in biochemistry. With bud $(157, 315)$ we have two unrelated "atoms" so to speak, and their manipulation in a few critical steps, similar to that between the few active sites of enzymes, "catalyzes" a relation, thus "bonding" them. □

5 Conclusion

It seems possible to "make mathematics ready" to expose critical vulnerabilities in the Syracuse problem. In this paper, we have already proven that the simple algebraic relation $Syr(4a + 1) = 4(Syr(a))$

produces non-trivial emerging behaviors in the Binary Tree. Firstly, it establishes the existence of a relation merging the orbits of alternate pairs of odd numbers along any odd branches of the tree. The existence of this relation implies that whoever can prove that for any odd number a, the orbits of $4a + 1$ and $S(4a + 1)$ merge solves Syracuse and that this be feasible we have called the Golden Gate conjecture. Secondly, we have demonstrated that a special, inevitable relation also merging their orbits exists between precisely defined pairs of odd numbers of rank 1, and the proof of this relation we have called the Banana-Split Theorem. Since any odd number of rank greater than 1 will be finitely transformed into an odd number of rank 1 under the Syr action, this result has some significance as it will manifest in the orbit of absolutely any odd number. We have also remarked that the forward orbit of any odd number of rank above 1 may only occupy red branches of type A, and that the forward orbit of any glacis number of order above 0 may only intersect a red branch number of either type B or C, and fly only through glacis of type B bottom.

Moreover, as whoever can successfully attack buds will successfully crack Syracuse, we may now outline a research program to expose vulnerabilities - whether decisive or not - in the buds of the Binary Tree; let us call it the "$GoldenProgram$", and split it in two scientific tasks that may be endeavored in parallel of each other, one regarding automatic theorem proving and deep learning, for example, in the vein of [2], and the other regarding a more analytic approach. Let us call the first program that of the "$GoldenAutomaton$", and the second one, the "$GoldenFormula$".

The first approach would consist of coding any family of "Golden Automata", able to solve any buds, namely, to demonstrate that for any two numbers forming a bud in the Binary Tree, their orbit can be proven to merge. Such automata could be assembled by mobilizing the current techniques of artificial evolution and deep learning and - which is an innovation per se - considering the Binary Tree as a self-calculating, infinite dataset.

The second approach may be defined more precisely, and will consist of solving and composing what we have called, in this paper, the B2G and G2B problems. Each of the solutions to these problems will imply the existence of a precise formula mapping branches to glacis, and glacis to branches respectively, the composition of which will provide us with a completely new understanding of the Syracuse orbits.

We shall contribute to the advancement of these two programs in a next paper.

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work and even a second author, a title to which nobody else may pretend.

In Memoriam

Solomon Feferman (1928-2016)

Alan Tower Waterman Jr. (1918-2008)

References

[1] Lagarias, J.ed *The Ultimate Challenge: The 3x+1 Problem.* American Mathematical Society 2010.

[2] Bowling, M; Burch, N; Johanson, M; Tammelin, O; *Heads-up limit hold'em poker is solved.* Science 09 Jan 2015:Vol. 347, Issue 6218, pp. 145-149