# Syracuse is solved

*complete proof v1.0*

Idriss J. Aberkane

12th November 2020

*Prepublished in an online lecture in French July 6th 2020*

**Abstract**

This self-contained communication details a complete solution of the Syracuse problem, based only on a theory, and the fundamental theorems it provided, that we published in 2017, but that are summarised here nevertheless. In a further explainer of the latter paper, which we published in 2020 for the interest of a group of non-experts, we gave a first complete algorithmic definition of a so-called "Golden Automaton", which had already been anticipated in our 2017 article and the structure of which rested strictly on this article's results. This updated publication is provided with an addendum formally demonstrating that the Golden Automaton's quiver can intersect any Syracuse orbit in finite time, thus demonstrating that any natural number finitely converges to 1 in the 3x+1 problem. The methodology of this final demonstration consists of identifying a "Syracuse determinant" that is, a collection of numbers that any diverging orbit must develop within, and demonstrating that the Golden Automaton's footprint on this determinant cannot allow the existence of any orbit it does not finitely intersect itself.

## 1 Introduction

In an open-access publication of 2017, we introduced a simple yet powerful methodology to study the Collatz orbits, essentially consisting of analysing each natural number with respect to its position on the complete binary, ternary and quaternary trees over $\mathbb{N}$, that is, trees defined respectively by the infinite iteration of all the possible compositions of the following linear operations on number 1:

- $\{\cdot 2; \cdot 2+1\}$ binary tree
- $\{\cdot 3; \cdot 3+1; \cdot 3+2\}$ ternary tree

- $\{\cdot 4; \cdot 4+1; \cdot 4+2; \cdot 4+3\}$ quaternary tree

...the binary tree thus generating an infinity of branches that has the cardinality of $\mathbb{R}$ (and the quaternary one, the cardinality of $\wp(\mathbb{R})$)[1]. For any number, defining its neighbourhood in terms of which branch (and of which length) it belongs to on each of these trees provides a framework to demonstrate very fruitful results that could actually be applied beyond the Syracuse problem, and more importantly, beyond discrete mathematics, for example in the study of the Julia sets of holomorphic functions. We have called the intersection of the binary, ternary and quaternary trees over N the "Romanesco (2,3,4)"

## 2 Definitions

**Note 2.1.** *For all intent and purpose we will define **Syr(x)** as **"the next odd number in the forward Collatz orbit of** $x$**"**.*

*Whenever two numbers $a$ and $b$ have a common number in their orbit, we will also note $a \equiv b$, a relation that is self-evidently transitive:*

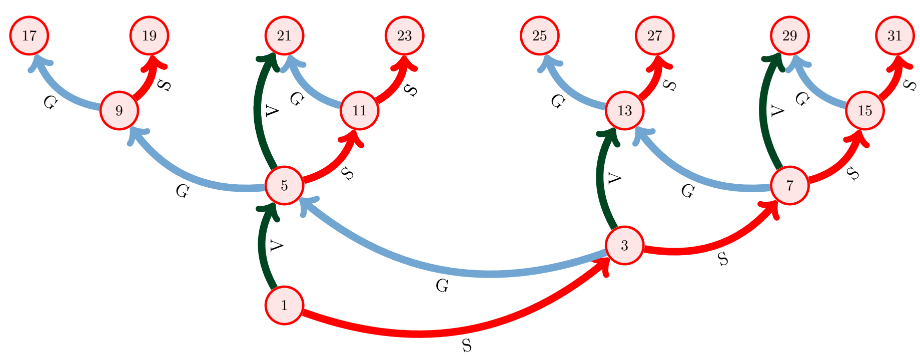
$\forall \{a; b; c\}$

$a \equiv b \wedge b \equiv c \to a \equiv c$

**Definition 2.1.** *Action S*
*The **Action S ("Successor")** on any natural number $a$ is defined as $S(a) = 2a + 1$*

**Definition 2.2.** *Action V*
*The **Action V ("Vertical")** on any natural number $a$ is defined as $V(a) = 4a + 1$*

**Definition 2.3.** *Action G*
*The **Action G ("Glacial")** on any natural number $a$ is defined as $G(a) = 2a - 1$*




[1] Interestingly enough, it is because of considerations regarding the cardinality of the set of all branches of the complete ternary tree over $\mathbb{N}$ that the founding observations leading to our work on Syracuse were made, initially over considerations regarding Feferman (2011)

**Figure 1**. Representation of operations V, G and S on all odd numbers from 1 to 31. The set $2\mathbb{N}+1$ is thus endowed with three unary operations without a general inverse and that are non commutative with $G \circ S = V$. Whenever we will mention the inverse of these operations, it will be assuming they exist on $\mathbb{N}$.

**Definition 2.4.** *Rank*
*The **rank** of an odd number is the number of **consecutive** end digits* 1 *in its base* 2 *representation, or equivalently, the number of times the action $S$ has been applied to generate it ($S$ is then defined on $\mathbb{N}$), and any number $o$ of rank* 1 *can be written $S(e)$ where $e$ is even.*

**Definition 2.5.** *Odd branch*
*An **odd branch** is the infinite set of numbers $\{a; S(a); S^2(a)\ldots\}$ where $a$ is of rank 1.*

**Definition 2.6.** *Glacis*
*The **glacis** of bottom $b$ is the infinite set of numbers $\{G(V(b)); G^2(V(b))\ldots\}$ where $b$ is any odd number.*

**Definition 2.7.** *Root*
*The **root** $r$ of any odd branch is its only number of rank 1.*

**Definition 2.8.** *Determinant*
*The **determinant** $d$ of any odd branch of root $r$ is $G^{-1}(r)$. Any number of rank* 1 *admits a determinant.*

**Definition 2.9.** *Vertical, Verticality*
*A number $a$ of rank* 1 *always admits a $V^{-1}(a)$ which is called its **Vertical**. To avoid any confusion, when ambiguous, we will call $V(a)$ the $V^+$ of $a$. If the vertical of $a$ is odd, we will call it **Vertical odd**, otherwise, it is **Vertical even**.*
*The **Verticality** of a number $a$ of rank* 1 *is the vector $\begin{bmatrix} n \\ b \end{bmatrix}$ where $b$ is either an even number or a number of rank* 2 *or more, and $a = V^n(b)$ We will say that $a$ has a verticality of $n$ and of bottom $b$.*

**Definition 2.10.** *Successal, Successality*
*A number $a$ of rank* 2 *or more will be called **Successal**, and its **successality** is equal to its rank.*

**Definition 2.11.** *Glaciality*
*In a glacis of bottom $b$, the **glaciality** of S(b) is set to* $-1$*, that of V(b) is set to* 0 *and that of $G^n(V(b))$ is set to $n$. To aggregate the information of the bottom $b$ of its glacis to any glacial number (that is, a number that can be written as $G(x)$ where $x$ is odd), its glaciality will be the vector $\begin{bmatrix} n \\ b \end{bmatrix}$*

# 3 Essential lemmas

**Lemma 3.1.** *If $a = V(b)$ and $b$ is odd, then $Syr(a) = (Syr(b))$ and we will note $a \equiv b$*

*Proof.* If $a$ is written $4b+1$ then $3a+1 = 12b+4 = 4(3b+1)$ therefore $a \equiv b$

□

This lemma is quite trivial and therefore in no way original, but it is an essential building block nonetheless.

**Lemma 3.2.** *Let $a$ be a number of rank 1 and of determinant $d$, then $Syr(S(a)) = G(3 \cdot d)$ Let $a$ be a number of rank $n$ in an odd branch of determinant $d$, then $Syr^{n-1}(a) = G(3^{n-1} \cdot d)$*

*Proof.* If $a$ is of determinant $d$ then $a = 2d - 1$, and of course $d$ is odd.

$S(a) = 4d - 1$

$\frac{3S(a)+1}{2} = \frac{12d-2}{2} = 6d - 1 = G(3 \cdot d)$

$Syr(S(a)) = G(3 \cdot d)$

Now let's generalize to the $n$

Note that if $Syr(S(a))$ can be written $G(3 \cdot d)$ it is also of rank 1, whereas S(a) was of rank 2, therefore, the action of Syracuse has made it lose one rank.

All we thus have to prove now is that $Syr(S^2(a)) = S(Syr(S(a)))$ under those conditions

$\frac{3 \cdot (S^2(a))+1}{2} = 6a + 5$

$S(Syr(S(a))) = S(3a+2) = 6a + 5 = Syr(S^2(a))$ □

**Corollary 3.3.** *If a is of rank $n > 1$, Syr(a) is of rank n-1, and Syr(S(a)) = S(Syr(a))*

**Note 3.4.** *The Syracuse action over an odd number is equivalent to adding 1 to it, then the half of the result, then $-1$. How many times one can add an half to the number $+1$ directly depends on the length of the immediate even branch of the binary tree that is to its right.*

*Let us take Mersenne numbers for example, that are defined as $2^n - 1$. One can Syracuse them consecutively a number of time that is proportionate to their rank-1, indeed, 31, which is written 11111 is of rank 5, because $32 = 2^5$ so if you repeat the action "add to* *the number the half of itself" which is equivalent to a multiplication by $\frac{3}{2}$ this will yield an even result* ***exactly four consecutive times****.*

*Thus, any ascending orbit in Syracuse concerns only numbers $a$ of rank 2 or more, and is defined by*

$$(a+1)\cdot(\tfrac{3}{2})^{rank-1}-1$$

*because the rank is strictly equivalent to the length of the next even branch of the binary tree on the right of the number, defining how many consecutive times the action $\cdot\frac{3}{2}$ will yield an even number.*

**Lemma 3.5.** *Let $a$ be an odd number of rank 1 that is vertical even, then $3a$ is successal, and $9a$ is vertical even.*
*Let $a$ be an odd number of rank 1 that is vertical odd, then $3a$ is successal, and $9a$ is vertical odd.*

*Proof.* If $a$ is vertical even it can be written $8k+1\ \forall k$

$3a = 24k+3$ and this number admits an $S^{-1}$ that is

$12k+1$, which is an odd number, therefore $3a$ is Successal

$9a = 72k+9$ and this number admits a $V^{-1}$ that is

$18k+2$, an even number indeed.

Now if $a$ is vertical odd, it can be written $8k+5\ \forall k$

$3a = 24k+15$ and $9a = 72k+45$, $3a$ admits an $S^{-1}$ and $9a$ admits a $V^{-1}$, respectively

$12k+7$ and $18k+11$ and they are both odd. □

**Theorem 3.6.** *(regular quaternary equivalence)*
*Let $a$ be a number that is vertical even, then $(a)\equiv S(a)$ and $S^k(a)\equiv S^{k+1}(a)$ for any even $k$. Let $a$ be a number that is vertical odd, then $S(a)\equiv S^2(a)$ and $S^k(a)\equiv S^{k+1}(a)$ for any odd $k$.*

*We will call these relations merging alternate pairs of odd branches* ***"regular quaternary equivalences"*** *or* ***qe****.*

*Proof.* If $a$ is vertical even then it can be written as $G(d)$ where $d$ is necessarily vertical (odd or even)

so by lemma 3.5 we have that $3d$ is successal and by lemma 3.2 we have $Syr(S(a)) = G(3d)$ so it is necessarily vertical odd (since $3d$ is successal) so $Syr(a) = V^{-1}(Syr(S(a))$ and therefore $a \equiv S(a)$

This behavior we can now generalize to the $n$, because if $a$ is vertical even and of determinant $d$, then the lemmas we used also provide that $Syr^n(S^n(a)) = G(3^n \cdot d)$ and therefore $Syr^n(S^n(a))$ will be vertical even for any even $n$ because $3^n \cdot d$ will be vertical something (even or odd, depending on what the determinant was) for any even $n$.

Now if $a$ is vertical odd it can be written $G(d)$ and $d$ is necessarily successal because $G \circ S = V$.

Thus $3d$ is vertical (even or odd) and therefore $Syr(S(a)) = G(3d)$ is vertical even. □

This **qe theorem** is a more elaborate, and now very useful building block of our demonstration, because it allows to place a relation of equivalence between every other pair of any odd branch of the binary tree, up to infinity. It is also based on the characteristics of increasing phases of the Syracuse orbits: any number of rank $n$ is finitely turned into a number that has a vertical, which is either odd or even.

We should now interest ourselves with the decreasing phases of the Syracuse orbits [2], and they only concern the glacis.

**Theorem 3.7.** *(glacial decreasing)*
*Let $a$ be a vertical even number with a glaciality of* $\begin{bmatrix} n \\ b \end{bmatrix}$ *where $n$ is even, then* $a \equiv 3^{\frac{n}{2}}(b)$
*Let $a$ be a vertical even number with a glaciality of* $\begin{bmatrix} m \\ b \end{bmatrix}$ *where $m$ is odd, then* $a \equiv S(3^{\frac{m+1}{2}}(b))$

*Proof.* if a is of glaciality $\begin{bmatrix} n \\ b \end{bmatrix}$

then by definition $a = 2^{n+2}b + 1$.

Then $3 \cdot a + 1 = 3(2^{n+2}b + 1) + 1) = 2^{n+2} \cdot (3b) + 4$.

As this expression can be divided by no more than 4, we have

$Syr(a) = 2^n 3b + 1$, therefore the glaciality of $Syr(a)$ is

$$\begin{bmatrix} n-2 \\ 3b \end{bmatrix}$$

Note that if $n = 2$ then $V^-(Syr(a)) = V^-(2^2 \cdot (3a) + 1) = 2^2 \cdot \frac{1}{4} \cdot (3b) = 3b$ which is of course an odd number. Therefore $Syr(a)$ is vertical odd and $V^-(Syr(a)) = 3b$ thus we have proven that $a \equiv 3b$.

[2]remember that we are still only considering odd numbers when we write "decreasing phases", and still defining $Syr(a)$ as "the next odd in the orbit of a"

If $n = 1$ then $a = 2^3 \cdot b + 1$ so $3(a+1) = 2^3 \cdot 3b + 4$ therefore $Syr(a) = S(3b)$ and thus $a \equiv S(3b)$.

From this we can generalise the progression of glacis numbers. Let $b$ be any odd number, thus defining a glacios bottom. All **"Variety S"** numbers of its glacis are written $V(b \cdot 2^{2k-1})$ or $S(b \cdot 2^{2k}) = 2^{2k+1} \cdot b + 1$ and all **"Variety V"** numbers of its glacis are written $V(b \cdot 4^k)$ or equivalently $S(b \cdot 2^{2k+1}) = 4^{k+1} \cdot b + 1$. Any glacis number $g$ of order $2k$ or $2k-1$ may be thus reduced to a glaciality of 0 or $-1$ by the following transformation: $(g-1) \cdot \left(\frac{3}{4}\right)^k + 1$ therefore we do have indeed that,

- for Variety S numbers : $2^{2k+1} \cdot b \cdot \left(\frac{3}{4}\right)^k + 1 = 2b \cdot 3^k + 1 = S(b \cdot 3^k)$
- for Variety V numbers : $4 \cdot 4^k \cdot b \cdot \left(\frac{3}{4}\right)^k + 1 = 4b \cdot 3^k + 1 = V(b \cdot 3^k)$

As the obtaining of these equalities will fit any odd number $b$, we have that any glacis number of glaciality $2k$ will be finitely mapped to $4b \cdot 3^k + 1 = V(b \cdot 3^k)$ and that any glacis number of order $2k-1$ will be mapped to $2b \cdot 3^k + 1 = S(b \cdot 3^k)$.. **Any glacis number merges either directly with a power of three of its bottom or with the successor of it.** □

# 4 General description of the Syracuse dynamic

The **qe theorem** allows to place an infinite amount of equivalences along the binary tree that are the result of quaternary properties.

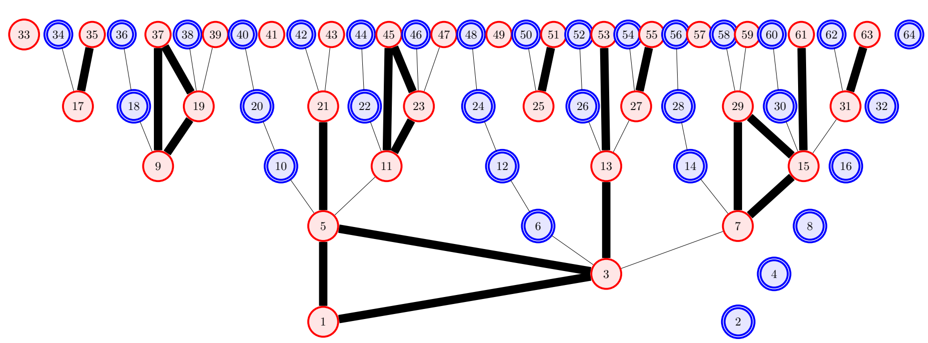


**Figure 2**. A representation of the intersection of the binary and the quaternary trees: as only the quaternary operation $\cdot 4 + 1$ really matters to the representation, we have just

warped the binary tree so that V(a) is indeed *Vertical* to $a$. We have also not connected the rank 1 numbers to the even ones of which they are the successors so as to make glacis more easily visible. The bold black lines indicate **qe** equivalences and vertical equivalences, and so whenever numbers are joined by a connective series of those lines, their Collatz orbits merge. Connecting all of those equivalences together completely solves the Syracuse problem, and as we will see in the next section, this endeavour requires the introduction of a third dimension: that of the ternary representation of any number.

## 4.1 Ascending phases

The orbit of an odd number can only increase if it is of rank 2 or more. Odd branch numbers of rank $n > 2$ ascend with the progression $+1 \cdot \frac{3}{2}^{n-1} - 1$ and this allows to compress their orbit to the next glacis.

More particularly though, if a pair is not connected by the **qe** equivalence, **it is because the rank 1 reduction of its smallest number by Syracuse is vertical odd**. When we have a pair $\{a; S(a)\}$ where $a$ is vertical odd, what happens is that $a$ is mapped to $Syr(V^{-1}(a))$ and $S(a)$ is mapped to the glacis of bottom $Syr(V^{-1}(a))$ and this phenomenon, by which a pair of numbers that were related on the binary tree become separated we call an **Avalanche**. Avalanches account for absolutely all of the chaoticity of the Syracuse orbits, and whoever can perfectly predict their occurence and consequences has cracked *all* of the Syracuse orbits. Needless to say, this communication doesn't have this ambition.

An example of Avalanche can be understood on **figure 2** by observing the pair $\{3; 7\}$ of which the first Syracuse image is $\{5; 11\}$ where 5 indeed is vertical odd. The avalanche is that 5 is mapped to the image of 1, which is 1, and 11 is mapped to 17, which is in the glacis of bottom 1. This happens to all such pairs, which we have called "buds" in a previous work.

The alacrious reader will not fail to notice that 17 is precisely a glacis number of Variety $V$, and this is not happening by chance: if a vertical odd number $a$ is the finite vertical of an even number, then $S(a)$ will be mapped as a variety $V$ in the next glacis, and proportionally as high as $a$ was vertical, and if the bottom of its verticality is odd, then $S(a)$ will be mapped as a variety $S$. The easy reason for which it is so will be left for the reader to grasp, as we do not use it in the other demonstrations of this piece.

We have also noted that any power of 9 of a vertical number is either of the series "vertical odd" or the series "vertical even" and these two are parallel: one cannot obtain a vertical odd number by applying any power of nine to a vertical even one. Thus, the destiny of branches the determinant of which falls within the "vertical odd" or "vertical even" series is quite different, and this level of precision can also help understand the Syracuse dynamic better. Still, we shan't use it further in this communication, even though it is more significant than a simple curiosity.

### 4.2 Descending phases

Any odd number can only decrease in *Syr* if it is of rank 1, and then it intersects either the Successor or the Vertical of a power of 3 of the bottom of its glacis. Note that in so doing, it always encounters the consequences of another **qe** on the branch it meets: either the power of three of the bottom of the glacis can be proven to merge with its successor, either it is vertical odd or it is its predecessor that is merging with it.

Here too, powers of 3 of the bottom of the glacis, just as being the case with odd branch determinants had useful implications to elaborate more advanced theorems, can either be of the "vertical odd" or "vertical even" sequences.

Numbers of even glaciality $n$ decrease following the dynamic $-1 \cdot (\frac{3}{4})^{\frac{n}{2}} + 1$ and those of odd glaciality $m$, $-1 \cdot (\frac{3}{4})^{\frac{m+1}{2}} + 1$.

Thus, **whenever one can prove that an odd number merges with its triple, one proves that it merges with its first number of even glaciality**

Also, **whenever one can prove that an odd number merges with the successor of its triple, one proves that it merges with its first number of odd glaciality.**

We cannot stress enough the importance of this pair of results.

# 5 Using the ternary tree to connect the quaternary equivalences

### 5.1 Definitions

In the previous sections we have mostly identified numbers by their position in the complete binary and quaternary trees over $\mathbb{N}$. Using statements of the kind "$a$ is vertical even" is a typical example of crossing binary and quaternary properties to identify specific characteristics of a number. We will now expand this methodology by adding the ternary tree, which elements we will identify with the following definitions:

**Definition 5.1.** *Ternary, Ternarity*
*A number $b$ is **ternary** or of **type B**, if it can be divided by 3. Its **ternarity** is the total number of times it can be divided by 3, to which we will add the information of the non-ternary number resulting from this finite operation, thus the full ternarity of any ternary number that can be written $3^n \cdot x$ where $x$ is non ternary is $\begin{bmatrix} n \\ x \end{bmatrix}$ For all intent and purpose, when we will refer to just the "ternarity" of a B type number (as opposed to "full ternarity") we will just be meaning $n$ alone.*

**Definition 5.2.** *1-ternary, 1-ternarity*
*A number $c$ is **1-ternary** or of **type C**, if its base 3 representation ends with digit 1. The number of times one can remove a consecutive end digit 1 (we call this operation $C^{-1}$) is its **1-ternarity**, to which we add the information of the number resulting from it. Thus, a number $c$ that can be written $x\underbrace{1\dots1}_{n}$ in base 3 has a 1-ternarity of $\begin{bmatrix} n \\ x \end{bmatrix}$*

**Definition 5.3.** *2-ternary, 2-ternarity*
*A number $a$ is **2-ternary** or of **type A**, if its base 3 representation ends with digit 2. The number of consecutive times one can remove an end digit 2 (we call this operation $A^{-1}$) is its **2-ternarity**, to which we add the number resulting from it. Thus, a number $a$ that can be written $x\underbrace{2\dots2}_{n}$ in base 3 has a 2-ternarity of $\begin{bmatrix} n \\ x \end{bmatrix}$*

**Definition 5.4.** *"up", "down"*

*A number is called **"up"** if its **qe** makes it merge with its successor. If $B \equiv S(B)$ we call it a **Bup** and respectively for $A$ and $C$, **Aup** and **Cup**. If $B \equiv S(B)$, as we necessarily have that $S(B)$ is of type $C$, we will call this $C$ "down" or $C down$. If a number is vertical odd, it is "down", if it is vertical even, it is "up".*

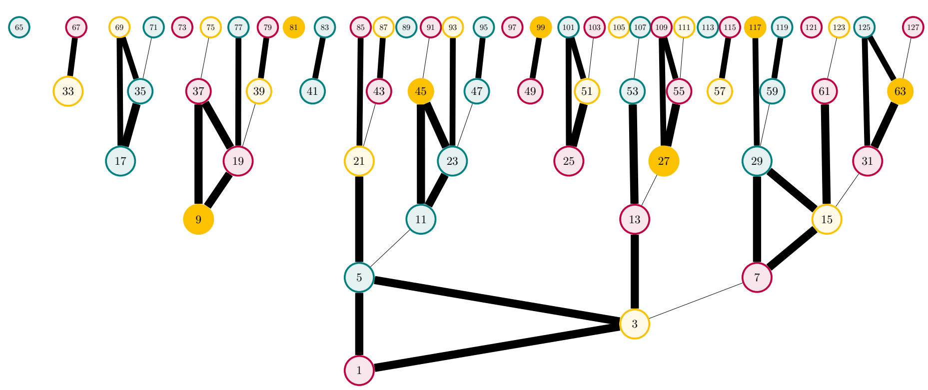



**Figure 3.** All odd numbers from $2^0$ to $2^7$. Type A are circled in teal, B in gold and C in purple. Numbers of a ternarity of 2 or more (numbers that can be divided by 9, that is) are also colored in gold. To explain again the previous definitions: 27 is a *Bup* and 63 is a *Bdown* for example. Any glacis number of glaciality above 0 is "up": 17 is an *Aup* for example, and 19 a *Cdown*. Though exotic, these names are absolutely essential to the results we obtain, and their very use is a pure result of our methodology: the "up" and "down" properties come from the study of the intersections of the binary and quaternary trees, and the A,B,C ones, from the ternary. This representation is a **Romanesco (2,3,4)**

## 5.2 Golden Automaton

We shall now define an algorithm that can mend together infinitely many quaternary equivalences, and actually **at least almost all of them**. This algorithm we call the **Golden Automaton** or **GA**.

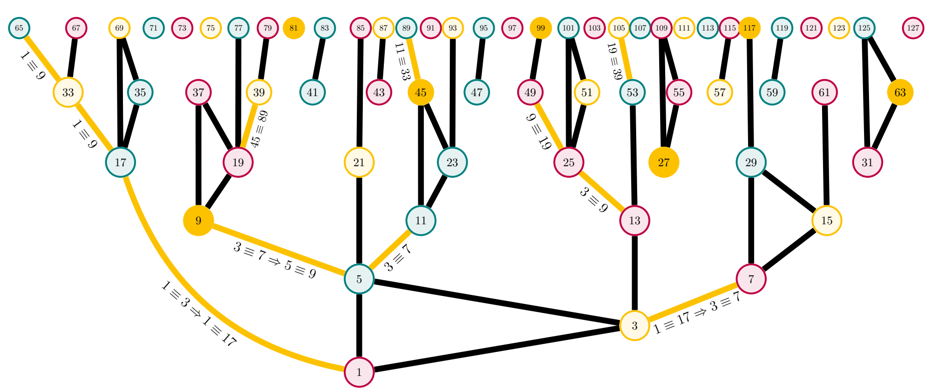
1. Start from the equivalence $1 \equiv 3 \equiv 5$
2. Whenever a new type B number is found in the network, divide it by 3, and always prioritize the Bups

   *for the first round, the only B is 3, so the GA notes that* $1 \equiv 3$
3. project the equivalences in the corresponding glacis

   *at round 1, there is only one equivalence: having that* $1$ *merges with its triple means it also merges with its first glacis number of the V Variety, thus giving* $17 \equiv 1$
4. whenever a new type A glacis number is grafter in the network, apply $Syr^{-1}$ up to a non A type[3], which is then added to the network. If this generates an equivalence of the type $B \equiv S(B)$, prioritize it.

   *at round 1, this gives* $17 \equiv 7$. *Also by the* ***qe*** $7 \equiv S(15)$ *and* $15$ *is of type B, but not a Bup. so it will not be prioritized by the GA. Still, we now have* $3 \equiv S(3)$, *which will be put to use in the glacis of bottom* $1$
5. project the equivalence in the corresponding glacis

   $3 \equiv S(3) \rightarrow 1 \equiv 9$. $9$ *is ternary, and being a glacis number, it is a Bup, so it will now prove two glacis equivalences together, namely those of* $49$ *and* $35$ *(each of them bringing two Bdowns) and equivalently,* $65$ *and* $33$.
6. After checking the *Aups* and prioritizing those of the highest 2-ternarity (they are easy to spot, as they can be written $G(B)$ where $B$ is of high ternarity) check all the *Bdowns* added to the network, including the ones that are the verticals of an $A$ type. Always prioritize the lowest numbers in any case.
7. repeat.




[3] This operation can only be finitely repeated of course, and is decreasing

**Figure 4.** The golden branches on this figure are only a *subset* (and precisely: the *Bup* series after exploiting 3) of the equivalences the GA proves to reach them: assuming number 3 already grafted to the network (since we start from $1 \equiv 3 \equiv 5$), they are the ones generated by only considering the *Bups*, and not the (many) more opportunities offered by connecting *Bdowns*, which are of lesser immediate interest but of much higher frequency and actually always yield either a new *Aup* or a new *Bup* to graft as well.

**Theorem 5.1.** *The Golden Automaton can neither stop nor loop.*

*Proof.* The moment a *Bup* is brought to its growing equivalence network, the GA can finitely prove that a number of either type $A$ or $C$ merges with two consecutive glacis numbers. Also, if the *Bup* is of a ternarity above 1, it will additionally prove that two glacial type $C$ merge with a type $B$ bottom, for example $1 \equiv 9$ means $65 \equiv 33 \equiv 1$ but also $49 \equiv 25 \equiv 3$ because 9 is of ternarity 2. Exploiting the first *Bup*, there will always be an *Aup* and *Bup* that will be freshly grafted to the equivalence network, either in the descending glacis order $(B; A)$ if the bottom is of type $C$, and $(A; B)$ if it is of type $A$. The more ternary the *Bup*, the higher-reaching the equivalence. That a *Bup* be brought in the network guarantees a full new equivalence of the type $x \equiv 3^k x \equiv S(3^k)$ which will always reconnect a new *Aup* and a new *Bup*. Thus, the grafting of a new *Bup* to the network always implies the grafting of another one. So it cannot stop.

Can it loop now? Any *Bup* the GA gets, it does by either operations $G \circ G \circ V$ or $G \circ V$, and then it will also get an *Aup* in the process. The equivalence it obtains is then that $\frac{Bup}{3}$ merges both with $G \circ G \circ V(\frac{Bup}{3})$ and $G \circ V(\frac{Bup}{3})$ this progression is strictly increasing, precisely because glacis numbers have a strictly decreasing progression in Syracuse. □

**Theorem 5.2.** *The Golden Automaton connects at least almost all orbits.*

*Proof.* So we have an algorithm that never stops, cannot loop and that recoups an exponentially growing diversity of numbers within the ternary tree over $\mathbb{N}$. Also, **all** the ascending Syracuse orbits lead to numbers that can be written $G(B)$, and **all** the descending ones, to numbers that are either $B$ or $S(B)$. Remember too, that, regarding the title of this communication, whenever an orbit is grafted by the GA, it is never *almost* bounded, but bounded, period.

Besides, we have not at all fully exploited all the *Bdown* the GA grafts to its equivalence network, and which new equivalences they also prove. Even though each of them is initially less powerful than the grafting a *Bup* (*ie.* in its first round of exploitation, a freshly grafted *Bdown* either offers the grafting of a new *Aup* or that of a new *Bup*) each round of running the GA brings proportionally much more of them, and this for simple reasons kind

- whenever the GA grafts a number of glaciality above 2, it also grafts its next glacial image, which is necessarily of type $C$. For example, as we have already seen, grafting 65 and 33 implies the grafting of the type $C$ 49 and 25 (their next glacial images) and therefore that of the *Bdowns* 99 and 51 which are their successors, which which they merge by virtue of the **qe**.

- whenever the GA grafts a type $A$ number, each application of $A^{-1}$ produces either another $A$ type which vertical is then a *Bdown*, or if it is of a 2-ternarity of 1, directly either a *Cup* or a *Bup*. As any glacial type $A$ can be written G(B), the ternarity of its determinant gives the number of times $A^{-1}$ can be applied (and so how many new *Bdowns* it gives, minus one). For example: 17 being the glacial image of 7, and $G(9)$, $A^{-1}$ can be applied only twice to it, so only one type $A$ number is also grafted (11 that is) thus grafting $V(11)$ which is 45.

Thus, *Bdown* that is grafted to the ternary tree (along with its third) always initiates the adding *at least* of an endless other series of *Bdowns* and of *Bups* too, this also, for elementary algebraic reasons:

- Any freshly grafted *Bdown* allows the grafting of either a glacis $A$ or a glacis $B$ depending on whether this *Bdown* can be written $3^n \cdot c$ (then it grafts an *Aup*) or $3^n \cdot a$ (then it grafts a *Bup*).
- if an *Aup* is thus grafted, it finitely connects either a *Bup* or a *Cup* and therefore a *Bdown* (as in what happens when connecting the pair $7 \equiv 15$), because this *Aup*, in base 3, can only be written either $b\underbrace{2\dots2}_{n}$ (in which case it finitely maps to a *Bup*) or $c\underbrace{2\dots2}_{n}$ (in which case it finitely maps to a *Cup*). Thus the play again through the *Bdown* series is guaranteed for the Golden Automaton.
- if it's a *Bup*, it now grafts *both* a glacis $A$ *and* a glacis $B$, the play again is also guaranteed.

Thus the GA does not stop, both through the grafting series that started with the first *Bup* (namely 9, then 33) but also to the one that started through the many *Bdowns* **after** it (15, 21, 45...). The existence of the **qe** also proves that the only problems that need to be solved to *fully* demonstrate that all Collatz orbits are bounded are the A-A avalanches, that is, the demonstration that all glacis numbers of type $A$ merge with their glacis Bottom, which would allow to prove that all the numbers of any odd branch merge. The Golden Automaton not only guarantees that an infinite amount of those A-A avalanches are solved but also that, for the progression of it to $3^n$, when n goes to infinity, *at least* **almost all** are solved. $\square$

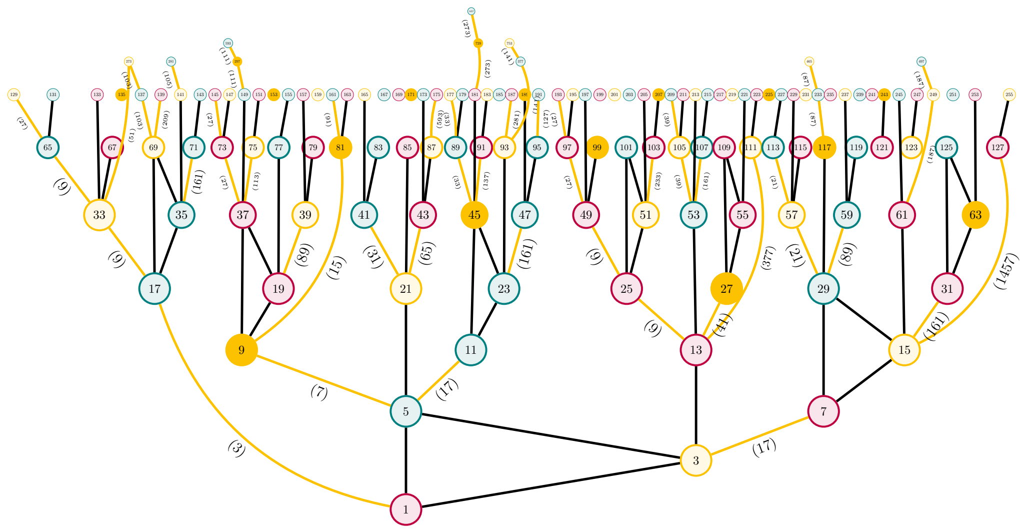


**Figure 5. Note that the development of the Golden Automaton mimics the one of the famous "Collatz Seaweed", which is of course no accident**. This more complete description of the Automaton, grafting the **qe** together as it progresses from $1 \equiv 3 \equiv 5$, goes all the way to grafting 127 and 31. Each branch indicates which number generates it, for example 9 is grafted to 5 because 7 is grafted to 3, and 9 being a *Bup* it grafts 33 and 65. **Never before has such a large proportion of any odd numbers**[4] **up to to $2^n + 5$ been demonstrated to finitely merge with 1, and with an algorithm that has of course nothing to do with the brute force computation of each orbits, in that not only are each of its steps finite and predictable, but the demonstrating that all numbers from $2^0$ to $2^n + 5$ converge has not the algorithmic complexity of calculating each single orbits**. In this figure are shown the grafting of 31 and 127, thus, even though all the consequences of this grafting are not represented, proving the merging of all numbers up to 261.

**The most problematic numbers at any stage $2^n$ remain the Mersenne numbers and of course, the ones leading to them**, yet the Golden Automaton grafts them very easily. All the problematic Mersennes are *Cups* like 31, 127 and the like. Since the determinant of the Mersenne branch is 1, at any level of the binary tree, their first glacial image will always be written $G(3^n)$. For example, the first forward glacial image of 31 is $G(81) = 161$. This is the reason 31, for example, cannot cycle when reaching its first glacis: being of high rank, it has to be written as the glacial of a proportionately high power of 3 of its determinant (1), which just can never be a vertical of 1 (no power of 3 of 1 can be written $V^n(1)$). Also, since multiplying a number of a high verticality by 3 always gives a number of proportionate successality, it means that the precursors of the Mersenne *Cups* will always be written $G(V^n(1))$, for example the precursor of 7 is 9 and that of 31 is $G(21) = 41$. Logically, the precursor of 127 is $G(85) = 169$, that of 511 is

[4]because it is easy to prove that if all numbers up to $2^n - 1$ converge then so do all odds until at least $2^n + 5$. Indeed, $2^n + 1$ being glacial, always have a smaller number than $2^n - 1$ in its orbit, from n=4 onward, $2^n + 3$ is always the successal of a glacial and $2^n + 5$ is always a vertical odd number

$G(341) = 611$.

Note that it is equally easy to demonstrate that the precursors of the "Anti-Mersenne" of type A, that is, the type A numbers that can be written $2^n + 1$ or equivalently the type A numbers of the glacis of bottom 1, can always be written $S(V^n(1))$: 11 leads to 17, 43 leads to 65, 171 leads to 257 etc.

As $729 = 3^6$ is of Variety S, $G(729) \equiv 127$ is of Variety V, which makes it much easier to solve as it brings its solving to the problem of grafting $91 \cdot 3 = 273$, which is itself of Variety V. This is why 127 is proven to merge with 15 before we even prove that 31 converges as well.

31 indeed is a much harder number to graft abstractly because its first glacial image is of Variety S, meaning it requires the grafting of many more ternary numbers and their successors to achieve its grafting, but it works nevertheless. It is easy to prove that if the first glacial image of 31 is of Variety S, then that of 127 is of Variety V, but that of 511 will be of variety S too. Hence: that 31 is "long" to solve means 127 is "short", and thus 511 is "long" but 2047 is "short". 31 was "long" also because 7 was "short": 9 is of variety S, hence the first glacial image of 7 is of variety V, and easy to graft back to 1.

One can very easily infer from **Figure 5** that with the infinity of avalanches the Golden Automaton solves, including those of the dreaded Mersenne numbers, *at least* ***almost all*** orbits attain, not *almost bounded* but **bounded** values, period. The emphasis on *at least* means it could be proven, through more abstract methods, that no Collatz Orbit can escape a finite development of the Golden Automaton, a statement this article will leave as a conjecture for now.

**Conjecture 5.3.** *For any Collatz orbit there is an arbitrarily long, but finite, development of the Golden Automaton that grafts it back to 1.*

# 6 Solving Syracuse

## 6.1 Fundamental rules of the Golden Automaton

To summarize, the Golden Automaton may be defined as the following set of formal arithmetic rules, which we may call **The Golden Rules**. They apply everywhere on $2\mathbb{N}+1$ but the Golden Automaton is defined as their application from number 3 onward. We shall call their application from any larger number $n$ onward, not *a priori* proven to merge with 1, a **Silver Automaton from** $n$. **As the reader has certainly understood already, our strategy consists of proving that the Golden Automaton intersects any Silver Automaton.**

1. $\forall x$ odd, $V(x) \equiv (x)$
2. $\forall x, k$ odd, $S^kV(x) \equiv S^{k+1}V(x)$ and $\forall x, k$ even, $S^kV(x) \equiv S^{k+1}V(x)$

3. $\forall n, y \in \mathbb{N}^2$, $\forall x$ odd non B, $3^n x \equiv y \Rightarrow \bigwedge_{i=1}^{n} (V(4^i 3^{n-i} x)) \wedge S(V(4^i 3^{n-i} x)) \equiv y$

4. $\forall n, y \in \mathbb{N}^2$ , $\forall x$ odd non B, $S(3^n x) \equiv y \Rightarrow \bigwedge_{i=1}^{n} (S(4^i 3^{n-i} x) \wedge S^2(4^i 3^{n-i} x)) \equiv y$

5. $\forall n \in \mathbb{N}$, $\forall y \in \mathbb{N}$, $\forall x$ odd non B where $3^n x$ is of rank 1, $a \equiv y$, $a = G(3^n x) \Rightarrow \bigwedge_{i=0}^{n} (S^i(G(3^{n-i} x)) \wedge S^{i+1}(G(3^{n-i} x))) \equiv y$

**Examples** Let us proceed with numbers 3 and 21

$3 = 3 \cdot 1$ therefore by **rule 3** we have $3 \equiv 17$

by **rules 1 and 2** we now have $3 \equiv 69 = 3 \cdot 23$ and $3 \equiv 141 = 3 \cdot 47$

also, $17 = G(3^2 \cdot 1)$ so by **rule 5** and **rule 1** we have $3 \equiv 45 \equiv 93 \equiv 7 \equiv 15$ and by **rule 4** we have $7 \equiv 9$ so starting by 3 only we have already proven the convergence of all odd numbers up to 31 with the exception of 27 and 31, but we already have by **rules 4 and 5** that $31 \equiv 27$ and we will also show that the Golden Automaton finitely proves that $15 \equiv 31$

We have $21 = V^2(1) = 3 \cdot 7$ so **rules 1, 2 and 3** give us that 113, 453, 227 and 909 converge while **rules 5 and 2** further demonstrate the convergence of 75 and $151 = S(75)$ and we keep going...

So any number than can be written $3^n a$ or $S(3^n c)$ solves a glacis B number and any number that can be written $3^n c$ or $S(3^n a)$ proves a glacis A number.

In turn, any glacis A number, which can always be written $G(3^n x)$ proves a cup or a bup, and on the way, strictly more than 2n type B numbers as well

We will call combining the fundamental rules of the Golden automaton ”Romanesco Arithmetic” **and it may be seen, interestingly, as a particular epistemological extension of modular arithmetic**.

Practicing Romanesco Arithmetic on number 15 for example finitely gives us numbers 31 and 27. Though the development of the Golden Automaton is branching, as each B type number that is vertical even provides us with both an A type and a B type number, we may follow only the pathway of type A glacis numbers to define a single non-branching arrow, which we may call ”pure A”. The latter leads us straight to 31, solving a great deal of other numbers on the way.

| | |
|---|---|
| $15 \equiv 81$ | glacial type B (the only one, as we follow a ”pure A” arrow from here) |
| $81 \equiv 1025$ | glacial type A |
| $1025 \equiv 303$ | branch Bup |
| $303 \equiv 809$ | glacial type A |
| $809 \equiv 159$ | branch Bup |

| | |
|---|---|
| $159 \equiv 425$ | glacial type A |
| $425 \equiv 283$ | branch Cup |
| $283 \equiv 377$ | glacial type A |
| $377 \equiv 111$ | branch Bup |
| $111 \equiv 593$ | glacial type A |
| $593 \equiv 175$ | branch Cup |
| $175 \equiv 233$ | glacial type A |
| $233 \equiv 103$ | branch Cup |
| $103 \equiv 137$ | glacial type A |
| $137 \equiv 91$ | branch Cup |
| $91 \equiv 161$ | glacial type A |
| $161 \equiv 31$ | branch Cup |
| $31 \equiv 41$ | glacial type A |
| $41 \equiv 27$ | branch Bup |

Again, it is in no way a problem, but rather a powerful property of the Golden Automaton that this particular segment of the infinite series of arrows (each being a *quiver branch*) already cover 19 steps (and actually more than that) because each of them is branching into other solutions. Simply put, any type A number in a glacis solves a Cup or a Bup, which in turn brings either directly another glacial type A or a glacial type B giving itself another glacial type B and a glacial type A.

The Golden Automaton thus generates an **infinite quiver** whose branches are infinite collections of arrows. Importantly, the rules of the Golden Automaton ensure that the set of its quiver's branches has a cardinality strictly above $2^{\mathbb{N}}$: intuitively, because it wraps itself diagonally around Romanesco (2,3,4) and because it has more branching points than the complete binary tree over $\mathbb{N}$, which has $\mathbb{R}$ branches already. **There are more branches of the Golden Quiver than there are real numbers**. This may also be proven easily by considering that each branch of the binary tree over $\mathbb{N}$ generates $\mathbb{N}$ unique vertical series, each composed of $\mathbb{N}$ many numbers,1/3 of them (the B) being the initial of a full (infinite) ternary tree. **The Golden Automaton is a diagonal *Quiver traversal* of the complete binary tree over $\mathbb{N}$ with a non-uniform branching factor of at least 3 at every node**. It is not a simple traversal sequence (e.g. the Kepler a.k.a Calkin-Wilf sequence) it is an infinite set of infinite trees, of which each single infinite branch can be mapped to one and only one of all the possible functions from $\mathbb{R}$ to $\mathbb{R}$.

Now note that the only problems on the way of solving Syracuse have always been **Cups** and **Bups**, because by induction, whoever could prove that any Cup or Bup has a lower number than itself in its orbit solves Syracuse. Furthermore, any Cup or Bup finitely maps to a Glacial A, therefore any glacial A that is grafted by the Golden Automaton solves a Cup (like 31) or a Bup (like 27). Besides, any Cup or Bup is finitely mapped to another Cup or Bup, because any glacial A is finitely mapped to either a B or a C type number by glacial decreasing.

The outline of our proof will consist of demonstrating that the Golden Quiver solves too many glacial A numbers - both denssely and diagonally - so too many Cups and

Bups to allow for any non-converging trajectory to exist over $\mathbb{N}$. Siply put, the Golden Quiver branches too much and covers too much, and since any supposedly non-converging trajectory will generate its own "Silver Quiver", our strategy consists of demonstrating that the footprint of any such silver quiver, for any supposedly non-converging Cup or Bup, grows too much not to intersect the Golden Quiver.

In turn the sets of Cups and Bups can be measured very easily in terms of its density over $\mathbb{N}$:

7 is the first Cup, 7+12 is a Cdown, 7+24 is a Cup, 7+36 is a Cup, 7+48 is a Cdown, etc. If we study the number crystal of C types of rank >1 (i.e. exactly this slice of Romanesco (2,3,4)) it gives

7 (U) - 19 (D) - 31 (U) - 43 (U) - 55 (D) - 67 (D) - 79 (D) - 91 (U) - 103 (U) - 115 (D) - 127 (D) - 139 (U) - 151 (D) - 163 (D) - 175 (U) - 187 (U) - 199 (U) - 211 (D) - 223 (D) ...

which is an **infinite braid** [5] made of infinitely many strands of the C of rank>1, with each same-rank strand of the braid following the rule $U \to D \to U$. As the same goes with the number crystal of B type of rank>1, simply put, for any 7 consecutive odd numbers in $\mathbb{N}$, a maximum of 3 are Cups and Bups and a minimum of 1 is, and as the binary tree develops, less than 16% of odd numbers are up, half of them being of a workable rank of 2 (like 27) meaning they are finitely mapped to another "up" of higher rank (e.g. 31 in the case of 27).

We may actually generalize the case of 27 in a theorem, to rigorously explain what we mean of a "workable" rank of 2, i.e that any up of rank 2 is finitely mapped to a higher rank:

**Theorem 6.1.** *Second order rank theorem*

*let $y = S(V(x))$ where $x$ is of rank $n$, then $y$ is finitely mapped to* $(y+5)\cdot(\frac{9}{8})^{\lfloor\frac{n}{2}} - 5$

*Proof.* Though rather esoteric, this little theorem is actually very useful, both in methodology and consequences. We have already seen that a founding theorem of the Romanesco algebra of the Syracuse problem was that any number $x$ of rank $n$ is finitely mapped to $(x+1)\cdot(\frac{3}{2})^n - 1$ which is of rank 1. In this theorem we can define the *second order rank* of a "bud" - ie. a number that can be written $S(V(x))$ where $x$ is odd - as the rank of $x$ itself, and we can demonstrate that any such bud will finitely decrease in second order rank and grow by increments of $\frac{9}{8}$ by being mapped to numbers of glaciality 1, then back to branches in a rank of 2.

For 27 we have indeed $(27+5)\cdot\frac{9}{8} - 5 = 31$

For 59 we have $(59+5)\cdot\frac{9}{8} - 5 = 67$

The general demonstration consists of observing that any number $y = S(V(x))$ where $x$

[5] Infinite braids of this kind are particularly interesting to study the Riemann Hypothesis, for example considering the infinite braid of multiplication tables over $\mathbb{N}$

is of rank 2 or more is mapped to $G^2(Syr(x))$ which can only be a number of glaciality 1, so it is mapped in turn to $S(3 \cdot S^{-1}(Syr(x)))$ thus giving our general formula

For a geometric intuition of the process, it could be summed up as "go to the closest number dividable by 8 on the right of the bud" (that's the +5) then compute one increasing $Syr$ (branch process) and one decreasing $Syr$ (glacis process), and that's the $\frac{9}{8}$, then give your +5 back. It is essentially the same process as computing $Syr^n$ of a rank $n$ odd number, only second order, in that it now concerns the bud right above said number.

□

As we have already noted any bud that is vertical even is strictly decreasing, that is, if a number can be written $S(V^2(e))$ where $e$ is even, then it maps to a type A number of glaciality 2, and therefore maps straight to $3 \cdot Syr(V(e))$ which is a strictly decreasing process.

So altogether, this eliminates all the Cups and Bups of rank 2 as problematic in the establishment of a final resolution of the Syracuse problem, so we now have at mosst 8% of odd numbers to worry about even before the Golden Automaton is taken into account, and actually, as we will see, much less than that.

## 6.2 Density and Diagonality of the Golden Quiver

**Rules 3, 4 and 5** ensure that the development of the Golden Automaton over the Binary tree is *diagonal*, which is absolutely essential. Finishing a proof of the Syracuse problem consists of demonstrating it is also *dense* enough not to allow any trajectory it does not capture. This would demonstrate at the same time that no cycles can exist in Syracuse but of course, that simply all natural numbers finitely converge to 1 in this discrete dynamical system.

As many chaoticians have observed since Lorenz, and chief among them Stuart Kauffman, chaos can actually remain quite "boxed" in its space phase, and this is exactly the case of the Syracuse dynamical system, which in spite of its deterministic chaoticity exhibits rather precise attractors within $\mathbb{N}$. However, if the Syracuse orbits converge to certain key numbers that can be beautifully visualised in the stems of the so-called "Collatz Seaweed", the Golden Quiver generates strictly more branches than there are real numbers and has therefore more branches than the complete binary tree over $\mathbb{N}$. In this current, and penultimate part of our demonstration, we will lay the foundations of this collision analysis to prove **Conjecture 5.3**

The way the Golden Automaton works, any type B and type C number proven to converges also provide either a Glacis A or a glacis B number, which in turn provide respectively a series of new B numbers and a branch Cup or Bup (in the case of a glacis A)

In a previous publication (2017) we demonstrated an interesting property of the Syracuse Dynamic we called the "Banana-Split Theorem"

**Theorem 6.2.** *Banana Split Theorem*

*Let $g_2$ and $g_1$ be glacis numbers of ranks 2 and 1 respectively then either*

- $g_2 \equiv g_1$ *("Vanilla")*
- $g_2 \equiv \frac{g_2-9}{8}$ *and* $g_1 \equiv (g_1 \cdot 8) + 9$ *("Banana")*

  *We will call the action $8x + 9$ "Banup" and the action $\frac{x-9}{8}$ "Bandown"*

*Proof.* By the **glacial decreasing theorem** (3.7) we have $g_2 \equiv 3b$ where $b$ is the bottom of the glacis. Then $b$ is either up or down. If it is up, $b \equiv S(b) \to g_2 \equiv g_1$. If it is down, $S(3b) \equiv S^2(3b)\ equiv V(S^2(3b))$ the latter being of type C, therefore $g_1 \equiv Banup(g_1)$. If $3b$ is down we also have either $3b = V(a)$ where a is of type A, or $3b = S(c)$ where c is of type C, either way it implies that $g_2 \equiv Bandown(g_2)$ □

It is also easy to demonstrate that any banana starting from a type A (and either going up or down) gives a type C number, and that any Banana from a type B gives another type B

This theorem, though anecdotal in appearance is actually very useful because it will guarantee that any trajectory merges with triple C or triple A numbers, which in turn is a an extremely important property of the Syracuse Dynamic. Indeed, it implies useful lemmas

**Lemma 6.3.**

- *Let $3{\cdot}a = V(x)$ where $x$ is odd and $a$ of type A, then $Banup(3{\cdot}a) = 3{\cdot}c$ where $c$ is of type C and the determinant of the branch of $x$ is dividable by 9.*

  *In plain English: any type A branch with a determinant dividable by 9, that is, a branch made of numbers that have been increasing at least twice under Syracuse has Triple A numbers above itself and triple C as any $g_2$ numbers in its glacis*

- *Let $V(a)$ where $a$ is of type A be dividable by 9, then $Banup(V(a))$ is dividable by 9 but neither is $V(S(a))$ or $Banup(V(S(a)))$. If V(a) is a triple B (dividable by 9) then V(S(a)) is a triple C and Banup(V(S(a)) is a triple A. If $a$ is of rank 2 or more, then $V(S(a) \equiv Banup(V(S(a)))$*

  *In plain English: let a Cup or Bup increase just once under Syracuse, then one will find triple As and triple Cs it merges with.*

*Proof.*

- It is trivial to demonstrate that by the Syracuse Dynamic, the action ·3 on any A branch gives a series of numbers that are vertical, except for the first number of the branch, for which the triple must be successal; also, since we have tripled an A branch (which determinant is necessarily of type B) the determinant of the second branch is dividable by 9. Now does $Banup(3a)$ give a $3a$ number? Yes it does $\frac{24a+9}{3} = 8a + 3$ which is of type A.

- What the second case meant in plain English is that if we apply the cation $\cdot 3$ on a CB branch now, we obtain a series of B numbers that are verticals to an A branch (again, except for the root of the CB branch). Then either these verticals are 3B numbers, and are dividable by 9, or they are triple Cs, and that is the reason one will find triple Cs above any A branch of which the determinant is dividable by 3 only once.

□

This allows us to further narrow the set of problems by formally defining it as, at most, a certain proper subset of the set of triple Cs that are Vertical odd, or a certain proper subset of the set of triple As that are Vertical odd. 69 is the first triple A that is vertical odd, and the next one, 141 which it happens to be equivalent to by rules 1 and 2, is found at a step of +72 over $\mathbb{N}$, so that we now have reduced the set of problems to below 2,8% of the odd numbers, evenly distributed of course over their set, and this without having even used the Golden Automaton from 3 of course. We may affirm that the set of actual problems is a proper subset of those less than 2,8% of odd numbers specifically **because any Cup or Bup of rank $n$ will, in its trajectory, intercept as many of those as its orbit inflates**.

Furthermore, one must also note that any Triple A or Triple C also constitutes the starting point of the grafting of infinitely many new odd numbers, so whenever a supposedly diverging Cup or Bup expands its footprint in the set of problems (which we later call a **Syracuse Determinant**) it does so exponentially. If its orbit inflates, it will intersect other Triple As and Cs but also other Cups and Bups forward, but each of the Triple As and Cs it intersects will themselves solve another infinite series of Cups and Bups, backward.

From a Ramsey-theoretical approach, it would be fascinating to define not some **Determinant** of Syracuse as we have still rather broadly done here, but an absolute **Critical Set** defined as the smallest set of odd numbers - if it exists - whose solving solves Syracuse. Such an endeavour is nevertheless not required at all to finish the proof, as the Golden Automaton simply solves way too many problems, and the problems' footprint (quiver) simply inflate way too much if we just assume they are problems (namely, that they dodge the Golden Automaton's own footprint), that a collision between the Golden Quiver and any supposedly non-connected Silver Quiver is inevitable anyway, thus proving **Conjecture 5.3**.

For any glacis B number we have an increasing and a decreasing pathway, which we saw in the example $15 \rightarrow 31$. Any Glacis B number finitely points to a pair of A and B glacis numbers, and, if it is dividable by 9 or more, to as many Bdown numbers as it is dividable by 3 minus one time. We now have that only Bups and Cups of rank strictly above 2 are problems in Syracuse, and that each time they grow they intersect twice as many triple As as their rank-1, plus one. Any number which orbit is supposed to escape the Golden Quiver will form an exponentially growing Silver Quiver within the space of Triple As. All we have to do now is to demonstrate that the Golden Rules starting from 3 grow too many solutions to avoid any possible Silver Quiver. This can be done in many ways but we would like to take the opportunity of this demonstration to expand upon the concept of Romanesco Calculus. Thus, although it could be demonstrated in a more straightforward manner, we will develop here a slightly more technical approach to build upon the more general interest of the methods and tools we have conceived in this article. We do not want to miss the opportunity to apply Romanesco algebra to a second order, that is, to now map all the triple A numbers to a 2-4 Romanesco and analyse their trajectory from this new referential. Again, this was not necessary in the first place, but the future potential of this sort of analysis warrants breaking ground for it in this very article already.

### 6.3 Romanesco analysis of the Syracuse Determinant

Romanesco analysis is a composition of rules, like Romanesco arithmetic, but on a dynamical system over a state space. In this final part of our proof, we demonstrate that the diagonality and density of the Golden Quiver interdicts any Silver Quiver defined by the property of not intersecting it.

Paramount to this demonstration is the analysis of the consequences of the Golden Rules, and in particular Rules 3, 4 and 5 implying that

- any triple A solves a glacial Bup
- any Bup solves a glacial A and a glacial Bup
- any glacial A solves a branch Cup or a branch Bup

but also, still by those same rules 3, 4 and 5, we may focus on the case of glacial Bup 81, which we have demonstrated finitely solves Mersenne 31. Since 81 is of ternarity 4, the Golden Rules imply it solves $2 \cdot (4-2) = 4$ triple Cs on its way to mapping to the single glacial A-B pair it solves, and any triple C solves a glacial A, which solves a branch Cup or Bup, and on its way, other triple As and triple Cs. The important part of this final step to solving the Syracuse problem is the analysis of the geometric consequences of action $\cdot 3$ on branches and glacis

- let $G_x = \{g_1 ... g_n\}$ be a glacis of bottom $x$, then $3 \cdot G_x = S(G_{3x})$ to which is added $3 \cdot g_1 = S(V(3x))$
- let $B_x = \{x, S(x) ... S^n(x)\}$ be a branch of root x, then $3 \cdot B_x = V(Syr(B_x))$ to which is added $G(Syr(S(x))$

These consequences of the Golden Rules ensure, among others, that any supposedly escaping orbit would also leave an exponentially growing quiver of triple Cs and triple As as a footprint. For example, the mere orbit of 7 also provides, among others, Triple A 483, (by banana between $241 = S^{-1}(483)$ and $29 = V(7)$, which had to be because of the rank of 7) but also triple A 753 and Triple C 93 by banana over $23 = S(Syr(7))$ and finally triple As 69 and 141 (over $17 = Syr^2(7)$) and triple C 1137 (by banana, again inevitable from the rank of 7 , from 1137 to 141)

A further step, though unnecessary *per se* to solve the Syracuse problem, consists of mapping all the triple A numbers that can be written as the vertical of an A type number and establish the Determinant of Syracuse within this space phase that any orbit has to intersect, also noting that any triple A provides a quiver of new solutions

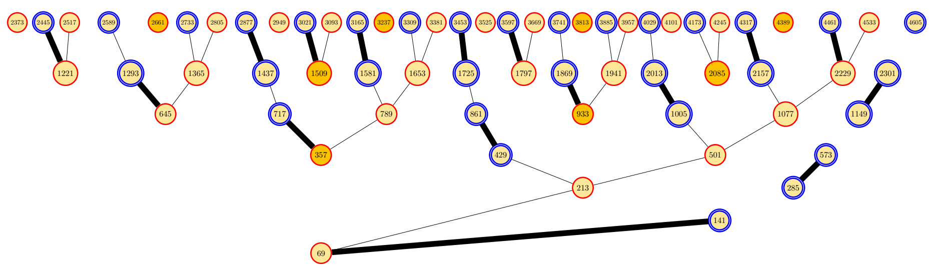



**Figure 6.** Representation of a Syracuse Determinant mapped onto a 2-4 Romanesco, made of the Triple A numbers that can be written $V(a)$ where $a$ is of type A. Each number is associated with a single integer, circled in blue if even, and red if odd. For any integer $n$, its associated Triple A is equal to $72n - 3$. The solid black lines plot the projection of the **quaternary equivalence** onto this representation. Only a subset of the Triple A numbers that are "vertical red" in this referential (i.e. that admit, in this representation, a $V^{-1}$ that is itself a Triple A of odd integer $n$ ; they are displayed in darker gold and they are +576 apart from each other in nominal value e.g. $357 + 576 = 933$) actually needs to be solved to solve the Syracuse problem and that the Golden Automaton solves an infinity of Triple A numbers in a density that is sufficient to interdict the existence of any other independent "silver quiver" solves the Syracuse Problem.

Simply put; the process generating problems cannot win against the process generating solutions, by virtue of its diagonality and the arities (plural, because they are not the same at every node) of its nodes, which are anywhere strictly above 2, and in fact, progressive ones indexed on the ternarity of the B nodes : taking the case of 81 as an example, we have this number of ternarity 4 mapping to 513 (glacial B) and 1025 (glacial A) but also, on its way, to triple A 771, triple B 1539, triple C 579, triple C 1155, triple C 867 and triple C 435, giving a final arity of 8 to this node, which depends only of the ternarity of 81. The Golden Quiver solves problems faster than they can appear, and each presumed problem grows a footprint that cannot escape it. Another approach to finish the proof consists of noting that there is a bijection from the set successal Cups and Bups to the set of glacial As, yet a proper surjection from the same set to the set of triple C numbers (each representing another glacial A of order 2) and that while only a proper subset of numbers of ternarity 1 needs to be solved, numbers of ternarity strictly above 1 solve exponentially more of those (e.g. case 81).

If any Silver Quiver is finitely intercepted by the Golden Quiver then not only cannot there be any kind of cycle in Syracuse, but every number also finitely converges to 1. Hence the peremptory title of this article: **Syracuse is solved**.

# 7 Conclusion

It may seem easy to quote theologian Lynn H. Hough's "life is a journey and not a destination" but it remains appropriate to state that this very work was a journey rather than a mere destination to solving the Syracuse Problem. The idea of studying morphisms of the complete binary tree over $\mathbb{N}$ came from discussions with Professor Solomon Feferman who invited the author as Visiting Scholar of Stanford University in 2006, and the purpose of such discussions was the Continuum Hypothesis, not the Syracuse Problem. If HC is independent from ZFC, the author's position was that it can only be "solved" by constructing an object with intermediate cardinality and demonstrating not that it can be forced upon the imagination, like the imaginary unit $i$ but that it can be useful to the working mathematicien. Thus was born Romanesco Algebra, which is essentially the tweaking of trees of various arities over $\mathbb{N}$ and more importantly, the study of some of their subsets (e.g. branches, glacis...) and also *sections* (e.g. braids of multiplication tables). In a similar fashion of conic sections, Romanesco sections bring about fascinating new objects of Modular and Diophantine arithmetic

Romanesco Algebra, and the use of Romanesco structures to map numbers should not be considered an obscure ad-hoc theory to attack - and ultimately solve - the Syracuse problem, but rather a larger epistemological contribution towards a full Diophantine Galois Theory, that is, the identification of algebraic structures that determine whether diophantine equations admit solutions or not. If studying the intersections of the complete binary, quaternary and ternary trees over $\mathbb{N}$ yields simple yet very powerful results regarding the Syracuse dynamic, as we have already pointed out this methodology could be extended to other discrete dynamical systems, diophantine problems and to other problems involving non-discrete environments.

## 7.1 The curious case of Marijn Heule

Among the few people whom the author contacted to share his early work is Marijn Heule from Carnegie-Mellon University, to whom he thoroughly presented his original Golden Automaton and some of its most important implications in Ramsey Theory. The author had indeed quoted Marijn Heule, describing his "Cube and Conquer" approach to satisfiability as both elegant and promising to finish one of the many possible proofs that the GA finitely proves the convergence of any orbit. Of course, Dr. Heule had a prior interest in Collatz, which he confirmed to the author during their many exchanges, by email and over the phone. Yet, after asking many rather through questions about the Golden Automaton he broke contact with the author only to announce in Qantas magazine that he had felt suddenly quite ”lucky” (sic) about attacking the Collatz conjecture again, and would probably yield tremendous new results on it. Some emails we shared have been mentioned on the youtube channel of the author, they remain at the disposal of any honest enquiry. Thus, after having thoroughly exchanged new concepts and techniques by email with the author, starting to storytell Qantas magazine of ”luck” is just... curious shall we say.

### 7.2 Testimonials

*In Memoriam*: Solomon Feferman (1928-2016), Alan Tower Waterman Jr. (1918-2008), John Conway (1937-2020).

This work received the help of Krystal Désirée Okanda and the attention of Prof. Pierre Collet. I was glad to share the beginning of this work to mathematician and youtuber Mickaël Launay during a skype meeting from San Francisco in the summer 2016.

IThe author solemnly testifies that the 2017 article had enough material to establish this complete proof of the Syracuse problem. However, had not some amateur mathematicians doubted of it, the author would probably not have written these extensions of the original theory and cannot but be grateful to them. The author salutes Sultanow et al. for the brilliant intuition that led them to study the Syracuse Problem with their angle on graph theory, though no form of interaction existed between the author and them before the publishing of this article.

This article is dedicated to the author’s infinitely patient and caring Mother, to the brave and persistent Paul Bourgine and Yves Burnod, to the playful and inspiring John Conway, and to the equally brave and courageous heir of anti-academic Diogenes, Nassim Nicholas Taleb.

## 8 Bibliography

1. ABERKANE, Idriss. *On the Syracuse conjecture over the binary tree.* 2017. hal-01574521

2. ABERKANE, Idriss. *L’intersection des arbres 2-3-4-aires complets sur N forme un repère et construit une solution au problème de Syracuse.* 2020. hal-02436242

3. BELAGA, Edward G. MIGNOTTE, Maurice. *Walking Cautiously into the Collatz Wilderness: Algorithmically, Number-Theoretically, Randomly.* 2006.

4. BOCART, Fabian. *Inflation propensity of Collatz orbits: a new proof-of-work for blockchain applications.* Journal of Risk and Financial Management, 2018, vol. 11, no 4, p. 83.

5. BOWLING, Michael, BURCH, Neil, JOHANSON, Michael, et al. *Heads-up limit hold'em poker is solved.* Science, 2015, vol. 347, no 6218, p. 145-149.

6. FEFERMAN, Solomon. *Is the continuum hypothesis a definite mathematical problem.* Draft of paper for the lecture to the Philosophy Dept., Harvard University, 2011, p. 2011-2012.

7. HEULE, Marijn JH, KULLMANN, Oliver MAREK, Victor W. *Solving and verifying the boolean pythagorean triples problem via cube-and-conquer.* International Conference on Theory and Applications of Satisfiability Testing. Springer, Cham, 2016. p. 228-245.

8. SULTANOW, E. KOCH, C COX, S. *Collatz Sequences in the Light of Graph Theory* Potsdam University 2020

9. TAO, Terence. *Almost all orbits of the Collatz map attain almost bounded values.* arXiv preprint arXiv:1909.03562 (2019).